Journal of Economic Theory **96**, 256–293 (2001)
doi:10.1006/jeth.2000.2642, available online at http://www.idealibrary.com on IDEAL®

Discounting and Long-Run Behavior: Global Bifurcation Analysis of a Family of Dynamical Systems[1]

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Received January 3, 2000; published online December 7, 2000

This paper is concerned with the relationship between the discount rate and the nature of long-run behavior in dynamic optimization models. The theory is developed under two conditions. The first is history independence, which rules out multiple limit sets. The second is a condition that avoids the reversion to a stable steady state, as the discount factor is lowered, once cycles have emerged. These conditions appear to be the minimal restrictions that would allow analysis by a bifurcation diagram. The results are illustrated by two well-known examples in this literature, due to Sutherland and Weitzman–Samuelson. *Journal of Economic Literature* Classification Numbers: C61, D90, 041. © 2001 Academic Press

Key Words: Discounting; long-run behavior; history independence; unique switching; bifurcation diagram.

1. INTRODUCTION

This paper is concerned with the following question: "How is long-run optimal behavior affected by changes in the rate at which the future is

[1] An earlier version of this paper was presented at a conference on Intertemporal Equilibrium Theory: Stability, Bifurcations and Indeterminacy at Meiji-Gakuin University in June, 1998. The current version has benefitted from the comments of Michele Boldrin and a referee.

0022-0531/01 $35.00

discounted?" Our objective is to try to answer this question[2] in terms of a *single* bifurcation diagram, which plots the relation between the *asymptotic behavior* of the *typical* optimal program and the discount factor.

There are two features about our exercise that are especially noteworthy. First, for the dynamical system generated by our dynamic optimization model, the law of motion is the optimal policy function. Even for very simple examples of dynamic optimization models, it is difficult to obtain the optimal policy function in closed form, and in general only a few basic properties of it are known. Nevertheless, we are able to obtain a fairly complete global bifurcation analysis for the family of dynamic optimization models as the discount factor is varied. This feature distinguishes our exercise from much of the mathematical literature, where the relevant law of motion is often known in closed form (for example, the quadratic family of maps), and a standard method of generating a bifurcation diagram is by iteration, on a computer, using this law of motion.[3]

Second, the class of examples that we study in detail, indicates an interesting feature about the transition from global asymptotic stability of optimal programs (to the stationary optimal stock) at high discount factors to global asymptotically stable cyclical behavior of optimal programs at lower discount factors. We do not always observe two-period cycles being "born," and gradually developing into cycles with larger amplitude as the discount factor falls. Instead, in some cases, two-period cycles appear past the bifurcation point, fully "grown up." Thus, in these cases, we are unlikely to observe "small cycles" on a sustained basis; loosely speaking, we are likely to observe *either* (approximately) stationary behavior *or* "large cycles."

There is a basic observation one can make regarding our stated objective. Since we would like a *single* bifurcation diagram to represent the relationship between discounting and long-run behavior, the dynamical system generated by our optimization model must be *history independent*, for each specification of the discount factor. That is, the asymptotic behavior of optimal programs from (almost) every initial stock must be independent of the initial stock itself.

To elaborate, let us consider the bifurcation diagram obtained in Fig. 1, for a variation of the Weitzman–Samuelson example. Here, a point on the graph $(\underline{\delta}; \underline{x})$ indicates that (given the transition possibility set, Ω, and the utility function, u, as fixed), if the discount factor is $\underline{\delta}$, and $(x_t)_0^\infty$ is the optimal program starting from the initial stock, x (in the state space $X = [0, 1]$), then for (almost) every x, x_t converges asymptotically to $\underline{x}$.

[2] This is a *comparative dynamics* question, in the terminology of Samuelson [28]. In comparative dynamics, we change a parameter and "we investigate the effect of this change on the whole motion or behavior over time of the economic process under consideration" (see Samuelson [28, Chap. 10, 11 and Appendix B]). The mathematical method of addressing such a question is now commonly known as *bifurcation analysis*.

[3] For a typical bifurcation diagram obtained in this way, see Collet and Eckmann [10].

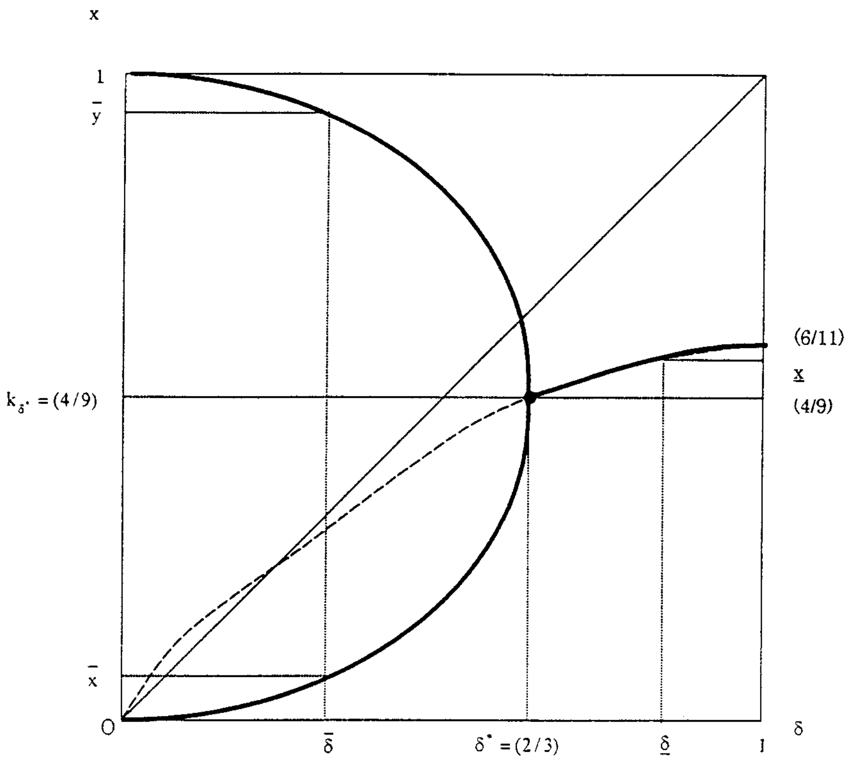

FIG. 1. Bifurcation diagram. Weitzman–Samuelson example: $\alpha+\beta<1$, $\alpha=0.54$, $\beta=0.45$.

The section of the graph, given by $(\bar{\delta}; \bar{x}, \bar{y})$ indicates that if the discount factor is $\bar{\delta}$, and $(x_t)_0^{\infty}$ is the optimal program starting from the initial stock x in X, then for (almost) every x, $(\bar{x}, \bar{y})$ is the set of limit points of $(x_t)_0^{\infty}$. [In particular, the section of the graph represented by $(\bar{\delta}; \bar{x}, \bar{y})$ *does not mean* that optimal programs from some initial stocks converge to $\bar{x}$, while optimal programs from some other initial stocks converge to $\bar{y}$].

In general, the dynamical system generated by our optimization model need not be history independent. For example, any dynamical system with multiple locally stable periodic points is necessarily history dependent. Thus, we need to impose some condition on our model in order to ensure history independent behavior.

The mathematical literature on dynamical systems achieves history independence by placing a restriction on the law of motion, known as the "negative Schwartzian derivative" condition. Since our law of motion is the optimal policy function, which is difficult to obtain in closed form in even very simple examples of dynamic optimization models, it is practically impossible to evaluate whether or not this condition is satisfied by our dynamical system. This basic problem differentiates our exercise from that appearing in the mathematical literature.

In the context of our special dynamic optimization model (which allows for period two cycles but no more complicated behavior than that), the device we employ is to place a condition (called Condition HI) on the optimal policy function of a two-period optimization model in which the terminal stock is restricted to be the same as the initial stock. This function can be computed, and it can be verified that it satisfies Condition HI for a class of examples, as we demonstrate in Sections 7 and 8 of the paper.

In order to complete our bifurcation analysis on the *family* of dynamical systems, generated by optimization models with different discount factors, we need to know how changes in the discount factor affect the local stability of the (unique) stationary optimal stock. We impose a condition on the model (see Condition US in Section 6) which ensures that the stationary optimal stock is locally stable for high discount factors, it loses its local stability when the discount factor gets lower than a critical level and never regains its local stability thereafter. (In general, of course, there might be several switches and reswitches between the two regimes). We verify in Sections 7 and 8 that Condition US is satisfied for a robust class of examples.

The family of examples in Section 7 constitute variations of the example of Weitzman, as discussed in Samuelson [29].[4] The class of examples in Section 8 constitute variations of the example of Sutherland [36].[5] Our theory, when applied to these examples, yields a global bifurcation diagram with the following features. There is a critical discount factor, $\hat{\delta}$, such that for $\delta > \hat{\delta}$, optimal programs exhibit global asymptotic stability to the (unique) stationary optimal stock, k_δ; and, for $\delta < \hat{\delta}$, optimal programs converge to a two-period cycle. For the example of Section 7 (Section 8), the amplitude of the two period cycle is seen to be monotonic (non-monotonic) in the discount factor.

We briefly indicate, now, the relationship of this paper to the available literature. The topic under discussion is best viewed as an exercise in trying to understand the relationship between a dynamic optimization model (specified by (Ω, u, δ)) and the optimal policy function generated by it. A basic question of interest in this area is whether the exercise of dynamic optimization imposes some restrictions on the nature of the policy function. Boldrin and Montrucchio [7] showed that any C^2 function can be a policy function of an appropriately defined dynamic optimization model.

[4] This example has been widely discussed in the literature; see, for example, Scheinkman [31], McKenzie [17], Benhabib and Nishimura [3].

[5] For discussions of Sutherland's example, see Cass and Shell [9], and Benhabib and Nishimura [3].

One implication of this finding was that optimal programs can exhibit chaotic dynamics, an observation that was verified by constructing suitable examples in the context of various economic models by (among others) Boldrin and Montrucchio [7], Deneckere and Pelikan [12], Majumdar and Mitra [15] and Nishimura and Yano [23].

It turned out, though, that many of these examples had associated with them, "unreasonably low" discount factors. A theory, examining the discount factor restrictions that must necessarily emerge in "'rationalizing" specific classes of chaotic policy functions by dynamic optimization models, was initiated by Sorger [33, 34]. For policy functions which generate period three cycles, an *exact* discount factor restriction was obtained independently by Mitra [20] and Nishimura and Yano [26].

Notwithstanding these discount factor restrictions (for specific classes of chaotic policy functions), it is possible to generate optimal programs which exhibit chaotic behavior when future utilities are discounted mildly, as demonstrated by Nishimura, Sorger and Yano [22], and Nishimura and Yano [24].

An aspect of the constructed examples in this literature, starting with the basic techniques of Boldrin-Montrucchio [7], is that the constructed utility function *depends on the chosen discount factor*. Thus, in a basic sense, this literature fails to address the question of how optimal behavior changes with changes in the discount factor, *given the transition possibility set*, Ω, *and the utility function*, u.[6]

This basic question has been investigated (at least partially) by several authors. It is known, of course, that the standard one-sector neoclassical model produces a degenerate bifurcation diagram as the discount factor is varied, since the unique non-trivial stationary optimal stock is globally asymptotically stable for all discount factors in $(0, 1)$. For specific classes of two-sector neoclassical models, Boldrin [5], Boldrin and Deneckere [6] and Nishimura and Yano [25] have studied changes in the nature of dynamic optimal behavior as the discount factor varies. In the context of the more general reduced-form model considered in this paper, Benhabib and Nishimura [3] provide an analysis of changes in the local stability behavior of the stationary optimal stock with changes in the discount factor.

The present investigation can be seen as a continuation of the above line of research. Its principal distinction is (i) in identifying two general sufficient conditions (*history independence* and *unique switching*) under which a satisfactory bifurcation analysis can be carried out in the context of the reduced-form model, which permits period-two cycles; and (ii) in demonstrating that

[6] This aspect was emphasized and brought to our attention in discussions with Lionel McKenzie.

these sufficient conditions can be verified in the context of two well-known classes of examples in this literature.

2. MATHEMATICAL PREREQUISITES

2a. *Dynamical Systems*

Let $X=[0,1]$ and g a map from X to X. We refer to X as the *state space*, and to g as the *law of motion* of the state variable $x \in X$. The pair (X, g) is called a *dynamical system*. Thus, if $x_t \in X$ is the state of the system in time period t, (where $t=0, 1, 2, ...$) then $x_{t+1}=g(x_t) \in X$ is the state of the system in time period $(t+1)$.

We write $g^0(x)=x$ and for any integer $t \geqslant 1$, $g^t(x)=g[g^{t-1}(x)]$. If $x \in X$, the sequence $\tau(x)=(g^t(x))_{t=0}^{\infty}$ is called the *trajectory* from (the initial condition) x. The *orbit* from x is the set $\gamma(x)=\{y\colon y=g^t(x)$ for some $t \geqslant 0\}$. The asymptotic behavior of a trajectory from x is described by the *limit set*, $\omega(x)$, which is defined as the set of all limit points of $\tau(x)$.

A point $x \in X$ is a *fixed point* of g if $g(x)=x$. A point $x \in X$ is called *periodic* if there is $t \geqslant 1$ such that $g^t(x)=x$. The smallest such t is called the *period* of x.

Note that if $x \in X$ is a periodic point, then $\omega(g^t(x))=\gamma(x)$ for every $t=0, 1, 2, ...$ A periodic point $\hat{x} \in X$ is *locally stable* if there is an open interval U (in X) containing $\hat{x}$, such that $\omega(x)=\gamma(\hat{x})$ for all $x \in U$. In this case, the periodic orbit $\gamma(\hat{x})$ is also called locally stable.

If g is continuously differentiable on X, and $\hat{x}$ is a periodic point of period t, then a *sufficient condition* for $\hat{x}$ to be locally stable is that $|Dg^t(\hat{x})|<1$. If $|Dg^t(\hat{x})|>1$, then $\hat{x}$ is not locally stable.

2b. *History Independence*

When we say that a dynamical system is history independent, we wish to convey the observation that the long-run (asymptotic) behavior of the state variable is independent of the initial state. While one might require such independence with respect to *every* initial state, it is more useful for many applications to insist on independence with respect to (Lebesgue) *almost every* initial state. Thus, a dynamical system will be history independent when the asymptotic behavior of the *typical trajectory* is independent of its initial state.

Formally, let (X, g) be a dynamical system and ν the Lebesgue measure on X. The dynamical system (X, g) is *history independent* if there is a subset E of X, such that for ν—almost every x in X, the limit set of x, $\omega(x)=E$. It is *history dependent* if it is not history independent.

2c. *Bifurcation Maps*

We will often be concerned with a *family* of dynamical systems, where the members of the family are described by a parameter. Formally, let us denote the parameter by $\mu \in P$, where P is taken to be a compact interval $[a, b]$ in $\mathbb{R}$, with $a < b$. A *family of dynamical systems* will then be denoted by (X, g_μ) where g_μ maps X to X for each $\mu \in P$.

Suppose the dynamical system (X, g_μ) is history independent for every $\mu \in P$. Then, for each μ, we can find a set $E(\mu)$, such that the limit set $\omega_\mu(x)$, for Lebesgue almost every x in X, is equal to $E(\mu)$. A *bifurcation map* is the correspondence which associates with each $\mu \in P$, its history independent limit set $E(\mu) \subset X$. A *bifurcation diagram* is a diagrammatic representation of the graph of the bifurcation map, (see Fig. 1).

3. THE MODEL

The model is described by a triple (Ω, u, δ), where Ω is the *transition possibility set*, u is the (period) *utility function*, and δ is the *discount factor*.

The following assumptions on (Ω, u, δ) will be maintained throughout the paper:

(A.1) $\Omega = X \times X$, where $X = [0, 1]$.

(A.2) $u: \Omega \to \mathbb{R}$ is a continuous function.

(A.3) u is concave on Ω, and if (x, z) and (x, z') belong to Ω, with $u(x, z) \neq u(x, z')$, then for every $0 < \lambda < 1$, we have $u(x, \lambda z + (1 - \lambda) z') > \lambda u(x, z) + (1 - \lambda) u(x, z')$.

(A.4) If $(x, z) \in \Omega$, and $x \leqslant x' \leqslant 1, 0 \leqslant z' \leqslant z$, then $u(x', z') \geqslant u(x, z)$. Also, $M \equiv \max_{(x, z) \in \Omega} u(x, z) > \min_{(x, z) \in \Omega} u(x, z) \equiv m$. Further defining $\Pi = \{(x, z) \in \Omega : u(x, z) > m\}$, (i) if $(x, z) \in \Pi$, and $x < x' \leqslant 1$, then $u(x', z) > u(x, z)$; (ii) if $(x, z) \in \Pi$, and $0 \leqslant z' < z$, then $u(x, z') > u(x, z)$.

(A.5) $0 < \delta < 1$.

A few remarks regarding the above assumptions are now in order. Assumption (A.1) simplifies the nature of transitions; it is possible to develop a theory with a more general transition possibility set, which is a convex, compact subset of $X \times X$, but it complicates the analysis without adding anything essential to the theory. Assumptions (A.2) and (A.5) are standard. Note that (A.3) ensures concavity of u, and a weaker form of strict concavity (in the second argument) than is commonly used. Similarly (A.4) ensures monotonicity of u, and strict monotonicity when the minimum utility is not attained. Together, (A.3) and (A.4) ensure that an optimal policy *function* exists in our framework.

A few basic implications of our assumptions can now be noted. First, we observe that

$$u(x, 0) > m \qquad \text{for all} \quad 0 < x \leqslant 1 \tag{3.1}$$

To see this, let (x^*, z^*) be such that $u(x^*, z^*) = M$. Then, by (A.4), $u(x^*, z^*) > m$, and $u(1, 0) > m$. Since $u(0, 0) \geqslant m$, for $0 < x \leqslant 1$, we have by (A.3), $u(x, 0) = u(x \cdot 1 + (1-x) \cdot 0, x \cdot 0 + (1-x) \cdot 0) \geqslant xu(1, 0) + (1-x)\, u(0, 0) \geqslant xu(1, 0) + (1-x)\, m > xm + (1-x)\, m = m$.

It follows from (3.1) that we have

$$u(x, z) > m \qquad \text{for all} \quad 0 < x \leqslant 1, 0 \leqslant z < 1 \tag{3.2}$$

For $0 < x \leqslant 1$, we have $u(x, 0) > m$ by (3.1); also, $u(x, 1) \geqslant m$. Thus, for $0 < x \leqslant 1, 0 \leqslant z < 1$, we have $u(x, z) = u(z \cdot x + (1-z) \cdot x, z \cdot 1 + (1-z) \cdot 0) \geqslant zu(x, 1) + (1-z)\, u(x, 0) > zm + (1-z)\, m = m$.

It follows from (3.2) that we have

$$u(x, z') > u(x, z) \qquad \text{for all} \quad 0 < x \leqslant 1, 0 \leqslant z' < z \leqslant 1 \tag{3.3}$$

To see this, notice that $u(x, z') > m$ by (3.2), while $u(x, z) \leqslant u(x, z')$ by (A.4). Thus, if (3.3) were not to hold, we would have $u(x, z) = u(x, z') > m$. But then by (A.4), we must have $u(x, z) < u(x, z')$, a contradiction, which establishes (3.3).

It also follows from (3.2) that

$$u(x', z) > u(x, z) \qquad \text{for all} \quad 0 \leqslant x < x' \leqslant 1, 0 \leqslant z < 1 \tag{3.4}$$

To see this, notice that $u(x', z) > m$ by (3.2), and by (A.4), $u(x', z) \geqslant u(x, z)$. Thus, if (3.4) were not to hold, then we would have $u(x, z) = u(x', z) > m$. But, then by (A.4), we must have $u(x', z) > u(x, z)$, a contradiction which establishes (3.4).

A *program* from $x \in X$ is a sequence $(x_t)_0^\infty$ satisfying

$$x_0 = x, \qquad \text{and} \qquad (x_t, x_{t+1}) \in \Omega \qquad \text{for} \quad t \geqslant 0$$

An *optimal program* from $x \in X$ is a program $(\hat{x})_0^\infty$ from x, such that

$$\sum_{t=0}^{\infty} \delta^t u(\hat{x}_t, \hat{x}_{t+1}) \geqslant \sum_{t=0}^{\infty} \delta^t u(x_t, x_{t+1})$$

for every program (x_t) from x.

4. VALUE AND POLICY FUNCTIONS

Under our maintained assumptions, there exists an optimal program from every $x \in X$. Thus, we can define a *value function*, $V: X \to \mathbb{R}$ by

$$V(x) = \sum_{t=0}^{\infty} \delta^t u(\hat{x}_t, \hat{x}_{t+1})$$

where $(\hat{x}_t)_0^\infty$ is an optimal program from x. Then, V is concave and continuous on X.[7]

It can be shown that for each $x \in X$, the Bellman equation

$$V(x) = \max_{z \in X} \{u(x, z) + \delta V(z)\}$$

holds. For each $x \in X$, we denote by $h(x)$ the set of $z \in X$ which maximize $\{u(x, z) + \delta V(z)\}$, among all $z \in X$. That is, for each $x \in X$

$$h(x) = \arg[\max_{z \in X} \{u(x, z) + \delta V(z)\}]$$

Then, a program $(x_t)_0^\infty$ from $x \in X$ is an optimal program from x if and only if $V(x_t) = u(x_t, x_{t+1}) + \delta V(x_{t+1})$ for $t \geqslant 0$; that is, if and only if $x_{t+1} \in h(x_t)$.

We call h the (optimal) policy correspondence. In order to demonstrate the existence of an optimal policy function, we have to show that $h(x)$ is a singleton for each $x \in X$.

Consider, first, the case $x > 0$. Suppose that z and z' both belong to $h(x)$, with $z > z'$. Thus (x_t) and (x'_t) are both optimal from x, with $x_1 = z$, $x'_1 = z'$. By (3.3) we have $u(x, z') > u(x, z)$. Then, defining $x''_t = \lambda x_t + (1 - \lambda) x'_t$ for $t \geqslant 0$, where $0 < \lambda < 1$, (x''_t) is a program from x. Further, $u(x''_0, x''_1) > \lambda u(x_0, x_1) + (1 - \lambda) u(x'_0, x'_1)$, and for $t \geqslant 1$, $u(x''_t, x''_{t+1}) \geqslant \lambda u(x_t, x_{t+1}) + (1 - \lambda) u(x'_t, x'_{t+1})$. Thus, we get $\sum_0^\infty \delta^t u(x''_t, x''_{t+1}) > \lambda V(x) + (1 - \lambda) V(x) = V(x)$, a contradiction to the definition of V.

Consider, next, the case $x = 0$. Suppose, as before, that z and z' both belong to $h(0)$, with $z > z'$. If $u(0, z) \neq u(0, z')$, the proof leading to a contradiction is the same as above. So, consider the case where $u(0, z) = u(0, z')$. Clearly, the common value cannot be greater than m, for then we must have $u(0, z') > u(0, z)$ by (A.4). Thus, $u(0, z) = u(0, z') = m$.

Let (x'_t) be optimal from $x = 0$, with $x'_1 = z'$. Then $(x, z, x'_2, x'_3, ...)$ is also optimal from x, since $u(x, z) = u(x, z')$, and $u(z, x'_2) \geqslant u(z', x'_2)$ by (A.4). By optimality of (x'_t), we must then have $u(z, x'_2) = u(z', x'_2)$. As before, this common value must equal m, otherwise using $z > z'$ and (A.4), we get a

[7] See, for example, Stokey, Lucas and Prescott [35], or Dutta and Mitra [13].

contradiction. By (3.2), then, we must have $x'_2 = 1$. Since (x'_t) is optimal, we get

$$(1+\delta)\, m + \delta^2 V(1) = u(0, z) + \delta u(z, 1) + \delta^2 V(1) = V(0) \tag{4.1}$$

The sequence $(0, 1, x'_3, x'_4, ...)$ is a program from $x = 0$, and so using (4.1),

$$m + \delta V(1) \leqslant u(0, 1) + \delta V(1) \leqslant (1+\delta)\, m + \delta^2 V(1) \tag{4.2}$$

This leads to the conclusion that

$$V(1) \leqslant m/(1-\delta) \tag{4.3}$$

But this clearly contradicts (3.1).

We have shown that h is a function from X to X. That is, for every $x \in X$, there is a unique optimal program from x. We refer to h as the (optimal) *policy function*. It can be shown that h is continuous on X.[8]

Following the analysis of Topkis [37], we know that the optimal policy function can be shown to be monotonic non-decreasing if the utility function is supermodular and monotonic non-increasing if the utility function is submodular.[9]

Since we want to allow the dynamical system (X, h) to exhibit cyclical behavior, we assume that the utility function is submodular. That is, we assume

(A.6) If x, x', z, z' belong to X, and $x' > x, z' > z$, then $u(x', z') + u(x, z) \leqslant u(x', z) + u(x, z')$

Under this additional assumption, it can be shown that h is monotonically non-increasing on X. Furthermore, there is a unique fixed point of the optimal policy function, h, which we call the stationary optimal stock.

5. HISTORY INDEPENDENCE

5a. *Background*

The dynamical system (X, h) generated by the dynamic optimization model (Ω, u, δ) is well-behaved in many respects. However, it is still difficult to construct a bifurcation diagram to indicate how the behavior of the dynamical system *changes* as the discount factor changes.

We show in this paper that for a *class* of dynamic optimization models, this task can be accomplished satisfactorily. This class is identified by two

[8] See, for example, Dutta and Mitra [13].

[9] See, also, the analysis in Benhabib and Nishimura [3].

conditions, which we refer to as history independence (Condition HI) and unique switching (Condition US). We examine the former condition in this section; the latter condition is taken up in Section 6.

To proceed with our analysis, we strengthen our basic set of assumptions on the dynamic optimization model as follows:

(A.2+) $u: \Omega \to \mathfrak{R}$ is a continuous function, and u is C^2 on the interior of Ω.

(A.3+) u is concave on Ω, and if (x, z) and (x, z') belong to Ω with $u(x, z) \neq u(x, z')$, then for every $0 < \lambda < 1$, we have $u(x, \lambda z + (1-\lambda) z') > \lambda u(x, z) + (1-\lambda) u(x, z')$. Further, for all (x, z) in int Ω, $u_{11}(x, z) < 0$, and $u_{11}(x, z)\, u_{22}(x, z) - [u_{12}(x, z)]^2 \geqslant 0$.

(A.4+) If $(x, z) \in \Omega$, and $1 \geqslant x' \geqslant x$, $0 \leqslant z' \leqslant z$, then $u(x', z') \geqslant u(x, z)$. Also $M \equiv \max_{(x, z) \in \Omega} u(x, z) > \min_{(x, z) \in \Omega} u(x, z) \equiv m$. Further, defining $\Pi = \{(x, z) \in \Omega : u(x, z) > m\}$, (i) If $(x, z) \in \Pi$, and $x < x' \leqslant 1$, then $u(x', z) > u(x, z)$; (ii) If $(x, z) \in \Pi$, and $0 \leqslant z' < z$, then $u(x, z') > u(x, z)$. And, for all (x, z) in int Ω, $u_1(x, z) > 0$, $u_2(x, z) < 0$.

(A.6+) If x, x', z, z' belong to X, and $x' > x$, $z' > z$, then $u(x', z') + u(x, z) \leqslant u(x', z) + u(x, z')$. Also, for all (x, z) in int Ω, $u_{12}(x, z) < 0$.

These strengthened assumptions reflect the C^2-differentiability of the utility function on the interior of the transition possibility set [(A.2+)], and the differential forms of concavity [(A.3+)], monotonicity [(A.4+)] and submodularity [(A.6+)] on that set.[10]

Under these strengthened assumptions, if $(x, h(x)) \in \text{int}\, \Omega$, then the Ramsey–Euler equation holds:

$$u_2(x, h(x)) + \delta u_1(h(x), h^2(x)) = 0 \tag{5.1}$$

Further, if $(x, h(x)) \in \text{int}\, \Omega$, and $x' > x$, then $h(x') < h(x)$. We verify this, first, for $1 > x' > x$. Since h is non-increasing, we have $h(x') \leqslant h(x)$. So, if the strict inequality did *not* hold, then $h(x') = h(x)$, and $(x', h(x')) \in \text{int}\, \Omega$. Thus, the following Ramsey–Euler equations must hold:

$$u_2(x, h(x)) + \delta u_1(h(x), h^2(x)) = 0; \; u_2(x', h(x)) + \delta u_1(h(x), h^2(x)) = 0$$

So, $u_2(x, h(x)) = u_2(x', h(x))$, which contradicts the fact that $u_{12}(x, z) < 0$ in int Ω. When $x' = 1$, $h(x') < h(x)$ follows from the result just established and the fact that h is non-increasing on X.

[10] The connection between the sign of the cross partial of the utility function and submodularity is explored in Ross [27] and Benhabib and Nishimura [3].

Our assumptions also ensure that for $x \in \operatorname{int} \Omega$, the ratio

$$\pi(x) = [-u_2(x, x)]/u_1(x, x)$$

is increasing in x. We now assume the following end-point condition:

(A.7) $\lim_{x \to 0} \pi(x) = 0$ and $\lim_{x \to 1} \pi(x) > 1$.

Then, given any $\delta \in (0, 1)$, there is a unique solution, k_δ, to the equation: $\pi(x) = \delta$. This k_δ is in the interior of X, and is the unique stationary optimal stock.

A useful property for the analysis in this section is a bound on the slope of the optimal policy function at the stationary optimal stock, k_δ, written as k below to ease the notation.

Note that, under our assumptions, if $(x, h(x)) \in \operatorname{int} \Omega$, then (by Theorem 1 of Benveniste and Scheinkman [4]) the value function V is differentiable at x, and $V'(x) = u_1(x, h(x))$.

Since $(k, k) \in \operatorname{int} \Omega$, we can find a neighborhood $N(k)$ of k, such that for all $x \in N(k)$, $(x, h(x))$ and $(h(x), h^2(x))$ are in the interior of Ω.

Let $x \in N(k)$, with $x \neq k$. To be precise, let $x < k$, so that $h(x) > k$ and $h^2(x) < k$. [The case $x > k$ can be handled similarly]. Then $V'(x) = u_1(x, h(x))$ and $\delta V'(h(x)) = \delta u_1(h(x), h^2(x)) = -u_2(x, h(x))$. Similarly $V'(k) = u_1(k, k)$ and $\delta V'(k) = \delta u_1(k, k) = -u_2(k, k)$. Thus, using concavity of u, we obtain the following two inequalities:

$$u(x, h(x)) + \delta V'(h(x))\, h(x) - V'(x)\, x \geqslant u(k, k) + \delta V'(h(x))\, k - V'(x)\, k$$

$$u(k, k) + \delta V'(k)\, k - V'(k)\, k \geqslant u(x, h(x)) + \delta V'(k)\, h(x) - V'(k)\, x$$

Adding the inequalities and transposing terms

$$\delta[V'(h(x)) - V'(k)][k - h(x)] \leqslant [V'(x) - V'(k)][k - x]$$

Iterating on this relationship, we get

$$\delta^2[V'(h^2(x)) - V'(k)][k - h^2(x)] \leqslant [V'(x) - V'(k)][k - x]$$

This yields the inequality

$$\delta^2[k - h^2(x)]/[k - x] \leqslant [V'(x) - V'(k)]/[V'(h^2(x)) - V'(k)]$$

We claim now that

$$\delta^2[k - h^2(x)]/[k - x] \leqslant 1 \tag{5.2}$$

For, if (5.2) were violated, we would have $[k - h^2(x)] > [k - x]/\delta^2 > [k - x]$. Thus, we must have $h^2(x) < x < k$, and $V'(h^2(x)) \geqslant V'(x) \geqslant V'(k)$, so that $[V'(x) - V'(k)]/[V'(h^2(x)) - V'(k)] \leqslant 1$. But then (5.2) must hold, a contradiction, which establishes (5.2).

5b. *A Sufficient Condition for History Independence*

A difficult step in our exercise is to devise a suitable sufficient condition for history independence. The mathematical literature provides a remarkably powerful condition for this purpose. The condition is that the dynamical system (X, g) satisfy g is C^3 and unimodal and the Schwartzian derivative of g is negative:

$$S(g(x)) = (g'''(x)/g'(x)) - (3/2)(g''(x)/g'(x))^2 < 0$$

at all points $x \in X$, where $g'(x) \neq 0$. [For the mathematical theory, see especially Singer [32], Guckenheimer [14], Misiurewicz [19] and DeMelo and van Strien [11]].

Unfortunately, this turns out to be not very useful for our purpose. One problem is that it is considerably difficult to ensure that the policy function, h, is C^1, and we know that higher-order differentiability of h does not hold typically (see Santos [30], Araujo [1] for a complete discussion of this issue). But this is probably not a major obstacle, because the negative Schwartzian derivative condition can be written without differentiability by using the concept of "cross-ratios" (see, for example, Guckenheimer [14]).

The more serious impediment appears to be the problem in assessing the *meaning* of this restriction on the policy function, even in a purely technical sense. This is because we do not typically know the policy function, but only a few of its basic properties, as discussed in Section 4. Even for simple examples of dynamic optimization models, we cannot, in most cases, obtain the policy function in closed form to verify whether or not it satisfies the negative Schwartzian derivative condition.

The device we use to get around these problems is to impose a condition, similar in spirit to the negative Schwartzian derivative condition, on the optimal policy function, f, of a two-period optimization problem, in which the terminal stock is restricted to be the same as the initial stock. The reason for focusing on the two-period optimization problem is that the fixed point of h coincides with the fixed point of f, and interior two-period cycles generated by h coincide with those generated by f.

Further, properties of f can readily be derived from the properties on the primitives of the model (Ω, u, δ). In particular, the condition on f which ensures history independence (see Condition HI in subsection 5d below) can be checked for robust classes of examples, as is demonstrated for variations on the examples of Weitzman–Samuelson [29] and Sutherland [36] in Sections 7 and 8 respectively.

5c. *On a Two-Period Optimization Problem*

For $x \in X$, consider the following optimization problem:

$$\left.\begin{array}{ll} \text{Max} & u(x, z) + \delta u(z, x) \\ \text{Subject to} & z \in X \end{array}\right\} (P)$$

For each x, problem (P) has a solution. Let $W(x)$ be the maximized value associated with (P).

It can be shown that for each $x \in X$, (P) has a *unique* solution. Suppose, on the contrary, that for some $x \in X$, z and z' both solve (P), where $z > z'$. If $x > 0$, then by (3.3), $u(x, z') > u(x, z)$. Thus, defining $z'' = \lambda z + (1 - \lambda) z'$, where $0 < \lambda < 1$, we get $u(x, z'') + \delta u(z'', x) = u(x, \lambda z + (1 - \lambda) z') + \delta u(\lambda z + (1 - \lambda) z', x) > [\lambda u(x, z) + (1 - \lambda) u(x; z')] + \delta[\lambda u(z, x) + (1 - \lambda) u(z', x)] = \lambda W(x) + (1 - \lambda) W(x) = W(x)$, a contradiction. If $x = 0$, then the above argument leads to a contradiction when $u(x, z') > u(x, z)$. And, when $u(x, z') = u(x, z)$, then we have $W(x) = u(x, z) + \delta u(z, x) = u(x, z') + \delta u(z, x) > u(x, z') + \delta u(z', x)$ [by (3.4)] $= W(x)$, a contradiction.

Let us denote the unique solution of (P), corresponding to $x \in X$, by $f(x)$. We also denote the second iterate of f (that is, f^2) by F.

It can be checked that (i) f is continuous on X; (ii) f is non-increasing on X, and if $(x, f(x))$ is in int Ω, and $x' > x$, then $f(x') < f(x)$. The continuity of f ((i) above) follows as usual by an application of the Maximum Theorem. The monotonicity properties of f ((ii) above) can be established by following the analysis used to establish the corresponding monotonicity properties of h.

If $(x, f(x))$ is in int Ω, then we have the first-order condition:

$$u_2(x, f(x)) + \delta u_1(f(x), x) = 0 \tag{5.3}$$

Since $u_{22}(x, f(x)) + \delta u_{11}(f(x), x) < 0$, we can use the implicit function theorem to conclude that f is differentiable at x, and $u_{21}(x, f(x)) + [u_{22}(x, f(x)) + \delta u_{11}(f(x), x)] f'(x) + \delta u_{12}(f(x), x) = 0$, which yields

$$f'(x) = -\frac{[u_{21}(x, f(x)) + \delta u_{12}(f(x), x)]}{[u_{22}(x, f(x)) + \delta u_{11}(f(x), x)]} \tag{5.4}$$

Thus, $f'(x) < 0$ whenever $(x, f(x)) \in \text{int } \Omega$.

5d. *Condition HI and Its Implications*

We are now ready to introduce Condition HI.

Condition HI (*history independence*). If *a, b, c are fixed points of F, and $a < b < c$, then* $F'(b) > 1$.

Our strategy in studying the implications of Condition HI (which will also clarify the meaning of the condition) is as follows. The magnitude of the derivative of f at the stationary optimal stock, k_δ, gives us information[11] on the magnitudes of the roots of the characteristic equation associated with the Ramsey–Euler equation (5.1) at $x = k_\delta$.

These roots in turn provide us with information[12] about the local stability (or instability) of the stationary optimal stock, k_δ, as a fixed point of the dynamical system (X, h).

Finally, Condition HI then ensures (almost) global asymptotic stability of the stationary optimal stock when it is locally stable. And, it ensures (almost) global asymptotic stability of a two-period cycle when the stationary optimal stock is locally unstable.

We now take up the formal analysis corresponding to each step of the above argument. For this purpose, it is convenient to break up our analysis into two cases, which we will call the "stable case" and the "unstable case."

The stable case: $0 < [-f'(k_\delta)] < 1$.

Step 1. We can write the characteristic equation [associated with the Ramsey–Euler equation (5.1) at $x = k_\delta$], as follows:

$$u_{12}(k, k) + [u_{22}(k, k) + \delta u_{11}(k, k)]\, \lambda + \delta u_{12}(k, k)\, \lambda^2 = 0 \tag{5.5}$$

where we denote k_δ by k to ease the notation.

Denoting the roots of the characteristic equation by λ_1 and λ_2 (where λ_1 has the least absolute value), we can infer that k is "saddle-point stable," that is

$$0 < (-\lambda_1) < 1 < (-\lambda_2) \tag{5.6}$$

To see this, observe first that by using (A.3+), (A.4+), λ_1 and λ_2 are real, and

$$\lambda_1 \lambda_2 = (1/\delta);\ (\lambda_1 + \lambda_2) = -\{[u_{22}(k, k) + \delta u_{11}(k, k)]/\delta u_{12}(k, k)\} \tag{5.7}$$

Thus, λ_1 and λ_2 are both negative, with $(-\lambda_2)^2 \geqslant (-\lambda_1)(-\lambda_2) = (1/\delta)$, so that $(-\lambda_2) > 1$.

Next, notice that (5.4) yields

$$[-f'(k)] = \frac{(1+\delta)[-u_{12}(k, k)]}{[-u_{22}(k, k)] + \delta[-u_{11}(k, k)]} \tag{5.8}$$

[11] Our method of analysis in Step 1 below, in both the "stable" and "unstable" cases, follows Benhabib and Nishimura [3] closely.

[12] The essential difficulty of this step arises from the fact that the policy function is not known to be differentiable at the stationary optimal stock. The sufficient conditions in the literature (see Araujo [1], Araujo and Scheinkman [2], Santos [30] and Montrucchio [21]) all involve a negative-definite Hessian of the utility function, an assumption we do not make.

Thus, in the "stable case" $[0<(-f'(k))<1]$, we have

$$(1+\delta)[-u_{12}(k,k)]<[-u_{22}(k,k)]+\delta[-u_{11}(k,k)] \tag{5.9}$$

Using (5.7) and (5.9), we then obtain $(1+\lambda_1)(1+\lambda_2)=1+\lambda_1+\lambda_2+\lambda_1\lambda_2 =[1/\delta(-u_{12}(k,k))][(1+\delta)(-u_{12}(k,k))-(-u_{22}(k,k))-\delta(-u_{11}(k,k))] <0$. Since $(-\lambda_2)>1$, we must have $(1+\lambda_1)>0$, and so $(-\lambda_1)<1$. This verifies (5.6).

Step 2. Let $N(k)$ be a neighborhood of k, such that for all $x \in N(k)$, $(x, h(x))$ and $(h(x), h^2(x)) \in \operatorname{int} \Omega$. For $x \in N(k)$, define

$$\mathbf{M}=\max\left[\limsup_{x\to k-}\left|\frac{h(x)-h(k)}{x-k}\right|, \limsup_{x\to k+}\left|\frac{h(x)-h(k)}{x-k}\right|\right].$$

By the analysis in the Appendix, $\mathbf{M}$ is either $(-\lambda_1)$ or $(-\lambda_2)$. We now show that $\mathbf{M}=(-\lambda_1)$.

By (A.1), we have

$$\begin{aligned}\tilde{u}_{21}+[(-\tilde{u}_{22})+\delta(-\bar{u}_{11})]\{[h(x)-h(k)]/(k-x)\}\\ -\delta(-\bar{u}_{12})\{[h^2(k)-h^2(x)]/(k-x)\}=0\end{aligned}$$

Using (5.2) we get

$$[(-\tilde{u}_{22})+\delta(-\bar{u}_{11})]\{[h(x)-h(k)]/(k-x)\}\leqslant(-\tilde{u}_{21})+[(-\bar{u}_{12})/\delta]$$

Thus, we obtain (by letting $x\to k$),

$$\begin{aligned}|Dh(k)|&\leqslant(1+\delta)[-u_{12}(k,k)]/\delta[(-u_{22}(k,k))+\delta(-u_{11}(k,k))]\\ &=[-f'(k)]/\delta\end{aligned}$$

for every Dini-derivative at k. Since $[-f'(k)]<1$, we get $|Dh(k)|<(1/\delta) <(-\lambda_2)$. Thus, $\mathbf{M}<(-\lambda_2)$ and consequently $\mathbf{M}=(-\lambda_1)<1$.

The above analysis implies that there is $\varepsilon>0$, such that for all $x\in(k, k+\varepsilon)$, $h^2(x)<x$ and for all $x\in(k-\varepsilon, k)$ $h^2(x)>x$.

Step 3. Since $[-f'(k)]<1$, there is $\varepsilon'>0$ such that for all $x\in (k, k+\varepsilon')$, $F(x)<x$ and for all $x\in(k-\varepsilon', k)$, $F(x)>x$.

Using Condition HI, we can infer that there is no $y\in(0,k)$ satisfying $F(y)=y$. Otherwise $y, k, f(y)$ are fixed points of F, with $y<k<f(y)$, and $F'(k)>1$, a contradiction. Similarly, there is no $y\in(k,1)$ satisfying $F(y)=y$. Thus, $F(x)<x$ for all $x\in(k,1)$ and $F(x)>x$ for all $x\in(0,k)$. (See Fig. 2 for an illustration).

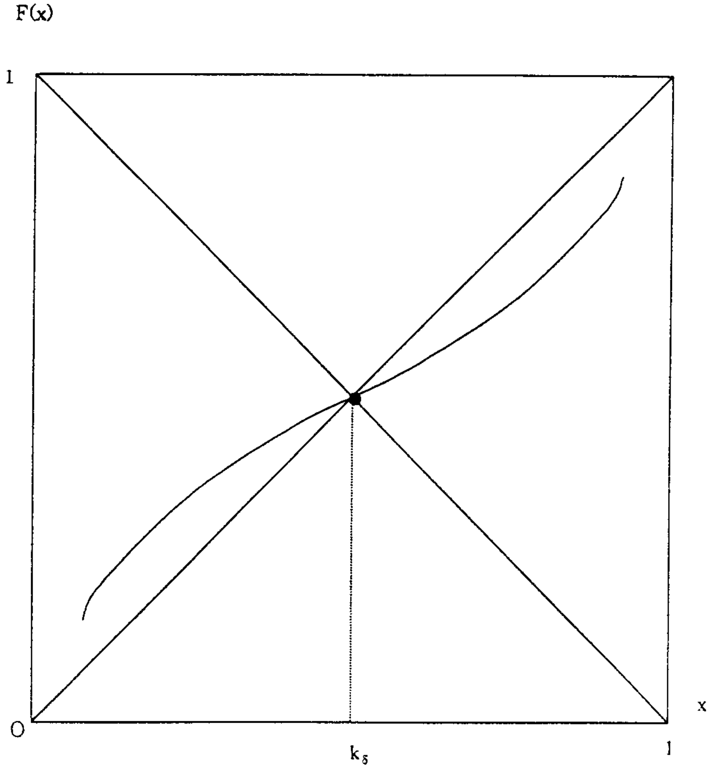

FIG. 2. Implication of condition HI (stable case): Diagram of $F = f^2$ for $1 > \delta > \delta^*$.

Since interior period-two points of h coincide with the interior period-two points of f, there is no $y \in (0, k) \cup (k, 1)$ satisfying $H(y) \equiv h^2(y) = y$. Given the conclusion of Step 2, we must, therefore, have $H(x) < x$ for all $x \in (k, 1)$ and $H(x) > x$ for all $x \in (0, k)$.

If $x \in (0, k)$, we must then have, along the optimal program (x_t) from x, x_{2t} increasing in t and converging to k, and x_{2t+1}, decreasing in t and converging to k. A similar statement can be made for $x \in (k, 1)$. Thus, for almost every $x \in X$, we have the optimal program (x_t) from x converging to k.

The Unstable Case: $[-f'(k_\delta)] > 1$.

Step 1. Denoting the roots of the characteristic equation (5.5) by λ_1 and λ_2 (where λ_1 has the least absolute value), we can infer that λ_1, λ_2 are real and negative and satisfy (5.7); further, k is "totally unstable;" that is

$$(-\lambda_2) \geqslant (-\lambda_1) > 1 \tag{5.10}$$

To show that $(-\lambda_1) > 1$, we use (5.8) and the information that $[-f'(k)] > 1$. This yields:

$$(1+\delta)[-u_{12}(k,k)] > [-u_{22}(k,k)] + \delta[-u_{11}(k,k)] \tag{5.11}$$

Using (5.7) and (5.11), we then obtain $(1+\lambda_1)(1+\lambda_2) = 1+\lambda_1+\lambda_2+\lambda_1\lambda_2 = [1/\delta(-u_{12}(k,k))][(1+\delta)(-u_{12}(k,k)) - (-u_{22}(k,k)) - \delta(-u_{11}(k,k))] > 0$. We have from (5.7) the information that $(-\lambda_2)^2 \geqslant (-\lambda_1)(-\lambda_2) = (1/\delta) > 1$, so that $(-\lambda_2) > 1$. Thus, we must have $(1+\lambda_1) < 0$, so that $(-\lambda_1) > 1$, verifying (5.10).

Step 2. Using the analysis of the Appendix, we know that $\mathbf{m}$ is either $(-\lambda_1)$ or $(-\lambda_2)$, where

$$\mathbf{m} \equiv \min\left[\liminf_{x\to k-}\left|\frac{h(x)-h(k)}{x-k}\right|, \liminf_{x\to k+}\left|\frac{h(x)-h(k)}{x-k}\right|\right].$$

Since $(-\lambda_2) \geqslant (-\lambda_1) > 1$, we have $\mathbf{m} > 1$. This establishes that there is $\varepsilon > 0$, such that for all $x \in (k, k+\varepsilon)$, $h^2(x) > x$, and for all $x \in (k-\varepsilon, k)$, $h^2(x) < x$.

Step 3. Since $[-f'(k)] > 1$, there is $\varepsilon' > 0$ such that for all $x \in (k, k+\varepsilon')$, $F(x) > x$ and for all $x \in (k-\varepsilon', k)$, $F(x) < x$.

Using Condition HI, we can infer that there is exactly one value of $y \in [0, k)$ satisfying $F(y) = y$. Otherwise, denoting by z the supremum of the fixed points of F, which are smaller than k, we have $F(z) = z$ and $z < k$, with $F'(z) \leqslant 1$. Then considering any fixed point $y < z$, we have a contradiction to Condition HI. Similarly, there is exactly one value of $z \in (k, 1]$ satisfying $F(z) = z$; namely $z = f(y)$. Call these values of y and z, y^* and z^*. Clearly $\{y^*, z^*\}$ is the unique period-two cycle of the dynamical system (X, f). (See Fig. 3 for an illustration).

Since interior period-two points of h coincide with the interior period-two points of f, we must have $H(x) < x$ for all $x \in (0, k)$ if $y^* = 0$, and $H(x) < x$ for all $x \in (y^*, k)$, $H(x) > x$ for all $x \in (0, y^*)$ if $y^* > 0$. Similarly, we must have $H(x) > x$ for all $x \in (k, 1)$ if $z^* = 1$, and $H(x) > x$ for all $x \in (k, z^*)$, $H(x) < x$ for all $x \in (z^*, 1)$ if $z^* < 1$.

If $x \in (0, k)$, we must then have, along the optimal program (x_t) from x, x_{2t} converging to y^* (monotonically decreasing if $x > y^*$, and monotonically increasing if $0 < x < y^*$). Also, x_{2t+1}, must converge to z^* (monotonically increasing if $x > y^*$, and monotonically decreasing if $0 < x < y^*$). A similar statement can be made for $x \in (k, 1)$. Thus, for almost every $x \in X$, we have the set of limit points of the optimal program (x_t) from x is $\{y^*, z^*\}$. Clearly $\{y^*, z^*\}$ is the unique period two-cycle of the dynamical system (X, h).

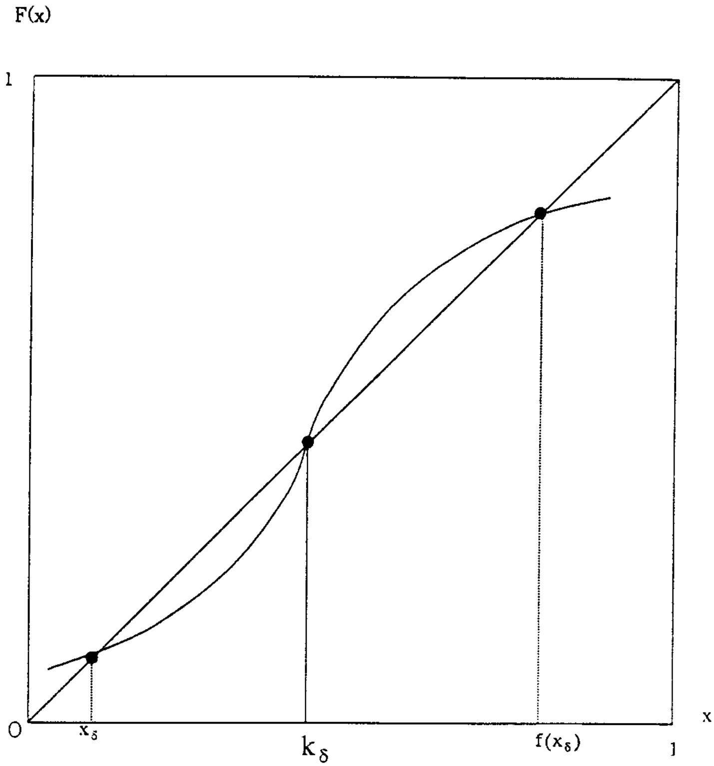

FIG. 3. Implication of condition HI (unstable case): Diagram of $F = f^2$ for $0 < \delta < \delta^*$.

6. DISCOUNTING AND THE STATIONARY OPTIMAL STOCK

Our objective is to see how the behavior of the dynamical system generated by the dynamic optimization model changes as the discount factor (of the dynamic optimization model) changes. In this section, we examine, as a basic step in this enquiry, how the stationary optimal stock, and its local stability property, are altered as the discount factor changes.

It is important, at this point, to recognize explicitly the fact that the policy function, and therefore its fixed point (the stationary optimal stock) depends on the discount factor. Henceforth, we consider (Ω, u) as fixed, and treat δ as a parameter, varying in (0,1). The dynamical system generated by (Ω, u, δ) is denoted by (X, h_δ) and the fixed point of h_δ by k_δ.

6a. *Monotonicity and Differentiability*

The stationary optimal stock, k_δ, can be shown to be monotonically increasing in the discount factor, δ. To see this, note that, for each

$0<\delta<1$, the stationary optimal stock, k_δ, satisfies $0<k_\delta<1$, and so by the Euler equation

$$[-u_2(k_\delta, k_\delta)]/u_1(k_\delta, k_\delta)=\delta \tag{6.1}$$

Let $0<\delta'<\delta''<1$, and suppose $k_{\delta'}\geqslant k_{\delta''}$. Then since $(-u_{21})$ and $(-u_{22})$ are positive on int Ω, $[-u_2(k_{\delta''}, k_{\delta''})]\geqslant[-u_2(k_{\delta'}, k_{\delta'})]$. And since u_{11} and u_{12} are negative on int Ω, we have $u_1(k_{\delta''}, k_{\delta''})\leqslant u_1(k_{\delta'}, k_{\delta'})$. Thus, we must have

$$[-u_2(k_{\delta''}, k_{\delta''})]/u_1(k_{\delta''}, k_{\delta''})\geqslant[-u_2(k_{\delta'}, k_{\delta'})]/u_1(k_{\delta'}, k_{\delta'}) \tag{6.2}$$

But the left hand side of (6.2) is δ'' and the right hand side of (6.2) is δ', so we must have $\delta''\geqslant\delta'$ a contradiction to the hypothesis that $\delta'<\delta''$.

To establish continuous differentiability of k_δ with respect to δ, define

$$J(k, \delta)=\{[-u_2(k, k)]/u_1(k, k)\}-\delta$$

for $k\in(0, 1)$ and $\delta\in(0, 1)$. Then, $J_1(k, \delta)=\{u_1(k, k)[-u_{21}(k, k)-u_{22}(k, k)] +[-u_2(k, k)][-u_{11}(k, k)-u_{12}(k, k)]\}/u_1(k, k)^2>0$. So, using $J(k_\delta, \delta)=0$, and the implicit function theorem, k_δ is continuously differentiable with respect to δ.

Two end-point properties of k_δ as a function of δ can also be established. Let us denote

$$\lim_{\delta\to 0} k_\delta\equiv k^0; \qquad \lim_{\delta\to 1} k_\delta\equiv k^1 \tag{6.3}$$

We can establish that $k^0=0$. For if $k^0>0$, then letting δ converge to zero in (6.1), we would get

$$\{[-u_2(k^0, k^0)]/u_1(k^0, k^0)\}=0 \tag{6.4}$$

But since $0<k^0<1$, $[-u_2(k^0, k^0)]>0$ and $u_1(k^0, k^0)>0$ by (A.4+), which contradicts (6.4).

We can also prove that $k^1<1$. For if $k^1=1$, then letting δ converge to 1 in (6.1), we get

$$\lim_{x\to 1}\pi(x)=1 \tag{6.5}$$

which contradicts (A.7).

6b. *Stability of the Stationary Optimal Stock under Mild Discounting*

Assumption (A.3+) ensures us that

$$\max[(-u_{11}(k^1, k^1)), (-u_{22}(k^1, k^1))]\geqslant(-u_{12}(k^1, k^1)).$$

We proceed now to assume that:

$$\text{(A.8)}\quad \max[(-u_{11}(k^1,k^1)),(-u_{22}(k^1,k^1))] > (-u_{12}(k^1,k^1)).$$

Notice that if $(-u_{11}(k^1,k^1))(-u_{22}(k^1,k^1)) > (-u_{12}(k^1,k^1))^2$ then (A.8) is clearly satisfied. Also, if $(-u_{11}(k^1,k^1))(-u_{22}(k^1,k^1)) = (-u_{12}(k^1,k^1))^2$, (A.8) is still satisfied, if $(-u_{11}(k^1,k^1)) \neq (-u_{22}(k^1,k^1))$.

Under (A.8), we have $[(-u_{11}(k^1,k^1)) + (-u_{22}(k^1,k^1))] > 2(-u_{12}(k^1,k^1))$. For δ close to 1, k_δ is close to k^1, and so $[(-u_{11}(k_\delta,k_\delta)) + (-u_{22}(k_\delta,k_\delta))] > (1+\delta)(-u_{12}(k_\delta,k_\delta))$. Thus, $[-f'(k_\delta)] < 1$, and we are in the "Stable Case". That is, k_δ is (almost) globally asymptotically stable.[13]

Beyond this, it is not possible in general to infer whether k_δ loses its local stability for some discount factors. If it loses its local stability (for δ not close to 1), it is also not possible in general to infer whether k_δ regains its local stability for still lower discount factors. It would appear that many patterns of behavior are possible.

6c. *Condition US* (*Unique Switching*)

In general, the set of discount factors for which local stability of the stationary optimal stock holds may be a complicated set to describe. Condition US ensures that there is a critical discount factor, δ^*, such that (i) for $\delta^* < \delta < 1$, the stationary optimal stock is locally stable, and (ii) for $0 < \delta < \delta^*$, the stationary optimal stock is locally unstable.

Condition US (*unique switching*). The function $R\colon (0,1) \to \Re_{++}$, defined by:

$$R(x) = \frac{[-u_2(x,x)][-u_{11}(x,x)] + u_1(x,x)[-u_{22}(x,x)]}{[u_1(x,x) + (-u_2(x,x))][-u_{12}(x,x)]}$$

satisfies $R(x) \downarrow R$ as $x \downarrow 0$ where $R \in (0,1)$.

Under (A.8), $R(k^1) > 1$, and so $R(k_\delta) > 1$ for δ close to 1. When $\delta \downarrow 0$, we know that $k_\delta \downarrow 0$, and so $R(k_\delta) \downarrow R \in (0,1)$. Thus, there is a unique $\delta^* \in (0,1)$, such that $R(k_{\delta^*}) = 1$. For $\delta^* < \delta < 1$, $R(k_\delta) > R(k_{\delta^*}) = 1$, and so $[-f'(k_\delta)] < 1$, which yields the "stable case." For $0 < \delta < \delta^*$, $R(k_\delta) < R(k_{\delta^*}) = 1$, so that $[-f'(k_\delta)] > 1$, which yields the "unstable case" (see Fig. 4). Thus, Condition HI and Condition US yield a complete bifurcation diagram of typical long-run optimal behavior as the discount factor varies.

[13] This is a very special case of a general result in turnpike theory under mild discounting. The large literature on this topic includes Brock and Scheinkman [8], Cass and Shell [9] and McKenzie [16, 18].

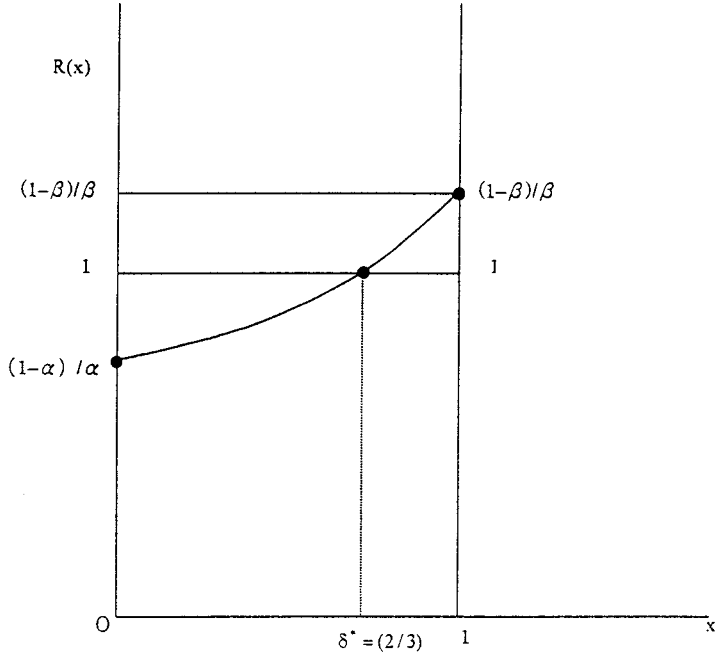

FIG. 4. Graph of $R(x)$ for Weitzman–Samuelson example: Condition US. $\alpha = 0.54$, $\beta = 0.45$.

7. VARIATIONS ON THE WEITZMAN–SAMUELSON EXAMPLE

We discuss in this section a variation on the example of Weitzman, as reported in Samuelson [29]. The utility function, $u\colon \Omega \to \Re$ is given by:

$$u(x, z) = x^{\alpha}(1-z)^{\beta} \qquad \text{where} \quad (\alpha, \beta) \gg 0, (\alpha + \beta) \leqslant 1, \alpha > 0.5 \tag{7.1}$$

Then, assumptions (A.1), (A.2+), (A.3+), (A.4+), (A.5), (A.6+), (A.7) and (A.8) are satisfied.

We now proceed to show that Condition HI is satisfied. It is easy to verify that, given any $\delta \in (0, 1)$, the unique stationary optimal stock, k_δ, is given by

$$k_\delta = \delta\alpha/(\beta + \delta\alpha) \tag{7.2}$$

We verify Condition HI as follows. We first show that if $F(x) = x$ has three solutions $0 < a < b < c < 1$, then the discount factor, δ, is smaller than a critical discount factor, δ^*; that is,

$$\delta < \delta^* \equiv \beta(2\alpha - 1)/\alpha(1 - 2\beta) \tag{7.3}$$

Next, we show that if b is also a fixed point of f, and $0<\delta<\delta^*$, then $[-f'(k_\delta)]>1$, so that

$$F'(k_\delta)=[F'(k_\delta)]^2>1 \tag{7.4}$$

To establish (7.3), suppose instead that $1>\delta\geqslant\delta^*$. If $0<x<1$ is any fixed point of F, then denoting $f(x)$ by y, we have the following two Ramsey–Euler equations:

$$\beta y^\alpha(1-x)^{\beta-1}=\delta\alpha x^{\alpha-1}(1-y)^\beta \tag{7.5}$$

$$\beta x^\alpha(1-y)^{\beta-1}=\delta\alpha y^{\alpha-1}(1-x)^\beta \tag{7.6}$$

Dividing (7.5) by (7.6), we obtain $(y/x)^\alpha(1-x)^{\beta-1}/(1-y)^{\beta-1}=(x/y)^{\alpha-1}(1-y)^\beta/(1-x)^\beta$, which can be simplified to: $(y/x)^{2\alpha-1}=(1-y)^{2\beta-1}/(1-x)^{2\beta-1}$. This yields:

$$y=x[(1-x)/(1-y)]^{(1-2\beta)/(2\alpha-1)} \tag{7.7}$$

Multiplying (7.5) by (7.6), we obtain

$$\beta^2xy=\delta^2\alpha^2(1-x)(1-y) \tag{7.8}$$

Substituting (7.7) in (7.8), we get $\beta^2x^2(1-x)^{2(1-\alpha-\beta)/(2\alpha-1)}=\delta^2\alpha^2(1-y)^{2(\alpha-\beta)/(2\alpha-1)}$, which yields:

$$(1-y)=(\beta/\delta\alpha)^{(2\alpha-1)/(\alpha-\beta)}x^{(2\alpha-1)/(\alpha-\beta)}(1-x)^{(1-\alpha-\beta)/(\alpha-\beta)} \tag{7.9}$$

Substituting (7.9) in (7.8), we get an equation involving only x:

$$\begin{aligned}&\beta^2x[1-(\beta/\delta\alpha)^{(2\alpha-1)/(\alpha-\beta)}x^{(2\alpha-1)/(\alpha-\beta)}(1-x)^{(1-\alpha-\beta)/(\alpha-\beta)}]\\&\quad=\delta^2\alpha^2(1-x)(\beta/\delta\alpha)^{(2\alpha-1)/(\alpha-\beta)}x^{(2\alpha-1)/(\alpha-\beta)}(1-x)^{(1-\alpha-\beta)/(\alpha-\beta)}\end{aligned} \tag{7.10}$$

Dividing (7.10) by x and rearranging terms, we get

$$\begin{aligned}&\beta^2-\beta^2(\beta/\delta\alpha)^{(2\alpha+1)/(\alpha-\beta)}x^{(2\alpha-1)/(\alpha-\beta)}(1-x)^{(1-\alpha-\beta)/(\alpha-\beta)}\\&\quad-\delta^2\alpha^2(\beta/\delta\alpha)^{(2\alpha-1)/(\alpha-\beta)}x^{(\alpha+\beta-1)/(\alpha-\beta)}(1-x)^{(1-2\beta)/(\alpha-\beta)}=0\end{aligned} \tag{7.11}$$

We denote the left-hand side of (7.11) by $G(x)$. Then $0<a<b<c<1$ are solutions of the equation $G(x)=0$. By Rolle's theorem, there exist $a<\bar{a}<b$ and $b<\bar{c}<c$ such that

$$G'(\bar{a})=0 \qquad \text{and} \qquad G'(\bar{c})=0 \tag{7.12}$$

That is, there are two distinct real roots of $G'(x)=0$ in (0,1).

We can differentiate $G(x)$ and simplify the expression to obtain

$$\begin{aligned} G'(x) &= [\delta^2\alpha^2/(\alpha-\beta)](\beta/\delta\alpha)^{(2\alpha-1)/(\alpha-\beta)} \\ &\quad \times [(1-x)/x]^2\,[(1-\alpha-\beta)(\beta/\delta\alpha)^2\,[x/(1-x)]^2 \\ &\quad -\{(\beta/\delta\alpha)^2\,(2\alpha-1)-(1-2\beta)\}[x/(1-x)]+(1-\alpha-\beta)] \end{aligned} \tag{7.13}$$

It follows that $\bar{a}$ and $\bar{c}$ are solutions of the equation

$$\begin{aligned} &(1-\alpha-\beta)(\beta/\delta\alpha)^2\,[x/(1-x)]^2-\{(\beta/\delta\alpha)^2\,(2\alpha-1)-(1-2\beta)\}[x/(1-x)] \\ &\quad +(1-\alpha-\beta)=0 \end{aligned} \tag{7.14}$$

Denoting $[\bar{a}/(1-\bar{a})]$ by A, $[\bar{c}/(1-\bar{c})]$ by B, we have $0<A<B$, and A and B are solutions of:

$$(1-\alpha-\beta)(\beta/\delta\alpha)^2\,z^2-\{(\beta/\delta\alpha)^2\,(2\alpha-1)-(1-2\beta)\}\;z+(1-\alpha-\beta)=0 \tag{7.15}$$

This implies $(\beta/\delta\alpha)^2\,(2\alpha-1)-(1-2\beta)>0$, and the discriminant of (7.15) must be positive, so that

$$(\beta/\delta\alpha)^2\,(2\alpha-1)-(1-2\beta)>2(1-\alpha-\beta)(\beta/\delta\alpha) \tag{7.16}$$

which can be rewritten as

$$(\beta/\delta\alpha)^2\,[(2\alpha-1)/(1-2\beta)]-1>2(\beta/\delta\alpha)(1-\alpha-\beta)/(1-2\beta) \tag{7.17}$$

Now, $(\beta/\delta\alpha)^2\,[(2\alpha-1)/(1-2\beta)]=(\beta/\delta\alpha)(\hat{\delta}/\delta)\leqslant(\beta/\delta\alpha)$. Thus, (7.17) yields

$$(\beta/\delta\alpha)[1-\{2(1-\alpha-\beta)/(1-2\beta)\}]>1 \tag{7.18}$$

But the left-hand side of (7.18) is $(\beta/\delta\alpha)[(1-2\beta-2+2\alpha+2\beta)/(1-2\beta)]$ $=(\beta/\delta\alpha)(2\alpha-1)/(1-2\beta)=(\delta^*/\delta)\leqslant 1$, a contradiction. This establishes (7.3).

Since $G'(x)=0$ has exactly two solutions ($\bar{a}$ and $\bar{c}$) a, b, c are the only fixed points of F. Since f is monotonic, b must be a fixed point of f, and $b=k_\delta$. We now proceed to establish (7.4).

For any $0<x<1$, we have $0<f(x)<1$, and the Euler equation:

$$\beta x^{\alpha}[1-f(x)]^{\beta-1}=\delta\alpha[f(x)]^{\alpha-1}(1-x)^{\beta}$$

is satisfied. This can be written as

$$(\beta/\delta\alpha)[x^{\alpha}/(1-x)^{\beta}]=[1-f(x)]^{1-\beta}/f(x)^{1-\alpha} \tag{7.19}$$

Differentiating (7.19) with respect to x,

$$\begin{aligned}(\beta/\delta\alpha)&[(1-x)^{\beta}\,\alpha x^{\alpha-1}+x^{\alpha}\beta(1-x)^{\beta-1}]/(1-x)^{2\beta}\\ &=\{f(x)^{1-\alpha}(1-\beta)[1-f(x)]^{-\beta}[-f'(x)]\\ &\quad-[1-f(x)]^{1-\beta}(1-\alpha)\,f(x)^{-\alpha}f'(x)\}/(f(x))^{2(1-\alpha)}\end{aligned} \tag{7.20}$$

Evaluating (7.20) at a fixed point, x, of f, we get

$$\begin{aligned}(\beta/\delta\alpha)&[(1-x)^{\beta-1}x^{\alpha-1}/(1-x)^{2\beta}][(1-x)\,\alpha+\beta x]\\ &=[-f'(x)][x^{-\alpha}(1-x)^{-\beta}/x^{2(1-\alpha)}][x(1-\beta)+(1-\alpha)(1-x)]\end{aligned} \tag{7.21}$$

Thus, at a fixed point, x, of f:

$$[-f'(x)]=(\beta/\delta\alpha)[(1-x)\,\alpha+\beta x]\,x/[x(1-\beta)+(1-\alpha)(1-x)](1-x) \tag{7.22}$$

Since the (unique) fixed point, $b=k_{\delta}$, of f, satisfies $b=\delta\alpha/(\beta+\delta\alpha)$, we have

$$(\beta/\delta\alpha)=(1-b)/b \tag{7.23}$$

Using this in (7.22), we get

$$[-f'(b)]=[(1-b)\,\alpha+\beta b]/[b(1-\beta)+(1-\alpha)(1-b)] \tag{7.24}$$

Now, $[(1-b)\,\alpha+\beta b]/[b(1-\beta)+(1-\alpha)(1-b)]=\{[(1-b)/b]\,\alpha+\beta\}/\{(1-\beta)+(1-\alpha)[(1-b)/b]\}=\{(\beta/\delta\alpha)\,\alpha+\beta\}/\{(1-\beta)+(1-\alpha)(\beta/\delta\alpha)\}=\{\beta(\alpha+\delta\alpha)\}/\{(1-\beta)\,\delta\alpha+(1-\alpha)\,\beta\}$. Thus, if $[-f'(b)]\leqslant 1$, we get $\beta\alpha+\beta\delta\alpha\leqslant\delta\alpha-\beta\delta\alpha+\beta-\alpha\beta$, which yields

$$\delta\geqslant[(2\alpha-1)\,\beta/\alpha(1-2\beta)] \tag{7.25}$$

Clearly (7.25) violates (7.3), so we must have $[-f'(b)]>1$, establishing (7.4).

Condition US can be verified in a more straightforward manner. The numerator of $R(x)$ is $\beta x^{\alpha}(1-x)^{\beta-1}\,\alpha(1-\alpha)\,x^{\alpha-2}(1-x)^{\beta}+\alpha x^{\alpha-1}(1-x)^{\beta}\,\beta(1-\beta)\,x^{\alpha}(1-x)^{\beta-2}$. The denominator of $R(x)$ is

$[\alpha x^{\alpha-1}(1-x)^{\beta}+\beta x^{\alpha}(1-x)^{\beta-1}]\ \alpha\beta x^{\alpha-1}(1-x)^{\beta-1}$. Thus, we obtain $R(x)$, after simplification, as:

$$R(x)=[(1-\alpha)+x(\alpha-\beta)]/[\alpha-x(\alpha-\beta)] \tag{7.26}$$

Since $\alpha>\beta$, as x decreases, we have the numerator of (7.26) decreasing and the denominator increasing, so $R(x)$ decreases. Further, we have $\lim_{x\to 0} R(x) =(1-\alpha)/\alpha<1$, since $1>\alpha>1/2$. This verifies Condition US.

We now examine two sub-cases of the above example in a bit more detail: (i) $\alpha+\beta=1$; (ii) $\alpha+\beta<1$.

Case 1 ($\alpha+\beta=1$). Here, the utility function is not strictly concave. However, (A.8) is still satisfied so long as $\alpha\neq(1/2)$. The critical discount factor, δ^*, and the corresponding stationary optimal stock, k_{δ^*}, are given by: $\delta^*=(\beta/\alpha)$; $k_{\delta^*}=(1/2)$.

For $\delta^*<\delta<1$, the stationary optimal stock, k_δ, is globally asymptotically stable. For $0<\delta<\delta^*$, optimal programs from all initial stocks other than k_δ converge to the period-two boundary cycle (0,1). At the bifurcation

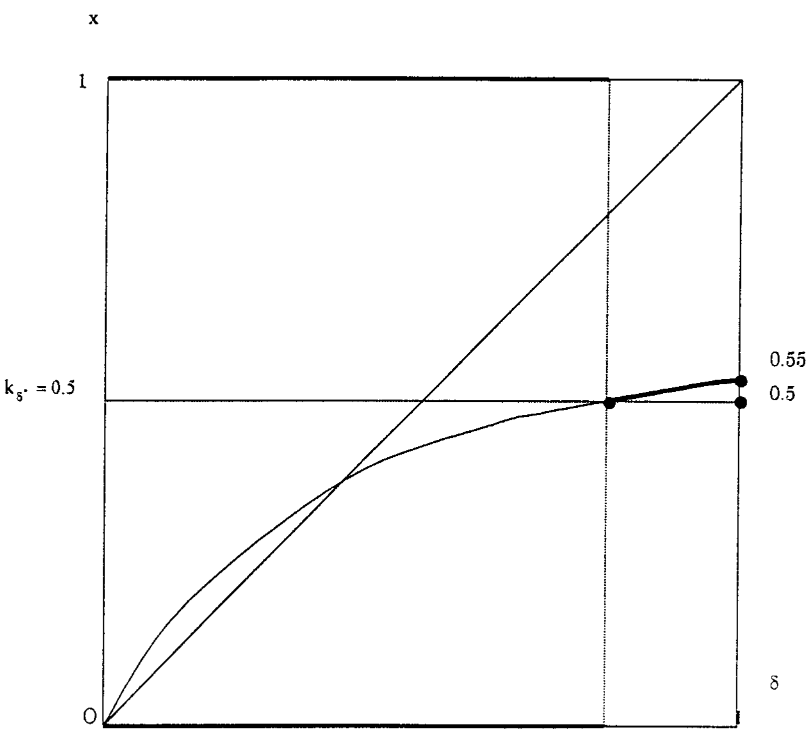

FIG. 5. Weitzman–Samuelson example: Bifurcation diagram: $\alpha+\beta=1$; $\alpha=0.55$.

point, δ^*, we have "neutral cycles:" starting from any initial stock, $x \in X$, the two-period cycle $(x, 1-x)$ is optimal. (See Fig. 5).

Weitzman's example (as reported in Samuelson (1973)) has $\alpha = \beta = (1/2)$. Thus, it is borderline in two ways: it satisfies $\alpha + \beta = 1$, and also $\alpha = \beta$. For every $0 < \delta < 1$, and for every x in X, the two-period cycle (x, y) is optimal, where $y = [\delta^2(1-x)]/[x + \delta^2(1-x)]$.

Case 2 $(\alpha + \delta < 1)$. Here, the utility function is strictly concave, and (A.8) clearly holds. The critical discount factor, δ^*, and the corresponding stationary optimal stock, k_{δ^*}, are given by:

$$\delta^* = \beta(2\alpha - 1)/\alpha(1 - 2\beta); \qquad k_{\delta^*} = (2\alpha - 1)/2(\alpha - \beta)$$

For $\delta^* < \delta < 1$, the stationary optimal stock, k_δ, is globally asymptotically stable.

For $0 < \delta < \delta^*$, optimal programs from all initial stocks other than k_δ converge to a unique period-two interior cycle. This interior cycle has "small" amplitude for δ close to δ^*. The amplitude increases as δ falls, and as δ converges to zero, this two-period cycle converges to (0,1). Thus, we obtain here the standard period-doubling flip bifurcation. (See Fig. 1.)

Notice that the theory developed in the paper ensures that we can obtain the precise bifurcation diagram of the family of dynamical system (X, h_δ), even though we do not know (and have not tried to estimate) h_δ for any of the δ-values.

8. VARIATIONS ON SUTHERLAND'S EXAMPLE

We discuss in this section a variation on the example of Sutherland [36]. The utility function, $u: \Omega \to \Re$ is given by:

$$u(x, z) = -ax^2 - bxz - cz^2 + dx \tag{8.1}$$

where $(a, b, c, d) \gg 0$, $4ac > b^2$, $b > 2c$, $2(a+b+c) > d > 2b(a-c)/(b-2c)$. Then, assumptions (A.1), (A.2+), (A.3+), (A.4+), (A.5), (A.6+), (A.7) and (A.8) are satisfied.

We can examine the solution to (P), given any $x \in X$, as follows. Since u is strictly concave on Ω, there is a unique solution, $f(x)$, to (P). Define $W(x, z) \equiv [u(x, z) + \delta u(z, x)]$, for (x, z) in Ω, and note that for each $x \in X$, $W_2(x, z) = u_2(x, z) + \delta u_1(z, x)$, so that

$$W_2(x, z) = -bx - 2cz + \delta[-2az - bx + d] \tag{8.2}$$

Given (8.2), there is a unique stationary optimal stock, k_δ, for each $0<\delta<1$, given by

$$k_\delta = \delta d/[(b+2c)+\delta(b+2a)] \tag{8.3}$$

Note that $(b+2c)+\delta(b+2a)>\delta(b+2c)+\delta(b+2a)=\delta(2a+2b+2c)>\delta d$, by assumption. Thus, we have $0<k_\delta<1$. Using (8.3), it is clear that k_δ is monotonically increasing in δ, with

$$\lim_{\delta\to 1} k_\delta = [d/2(a+b+c)]; \qquad \lim_{\delta\to 0} k_\delta = 0 \tag{8.4}$$

We now proceed to verify Condition HI. Define a critical discount factor, δ^*, as

$$\delta^* = (b-2c)/(2a-b) \tag{8.5}$$

Note that $\delta^*>0$, since $2a>b>2c$. Also, $(2a-b)-(b-2c)=2a+2c-2b>0$, so $\delta^*<1$.

We first show that if $F(x)=x$ has three solutions $0\leqslant a<b<c\leqslant 1$, and $\delta\neq\delta^*$, then it must be the case that $0<\delta<\delta^*$, and b must be the fixed point of f. Next, we show that if $0<\delta<\delta^*$, the fixed point of f, k_δ, satisfies $[-f'(k_\delta)]>1$.

To establish the first step, suppose instead that $1>\delta>\delta^*$. Define

$$\phi(x) = [\delta d/2(a\delta+c)] - [b(1+\delta)/2(a\delta+c)]\,x \tag{8.6}$$

for $x\in X$. We observe that:

$$\phi(k_\delta)=k_\delta; \qquad \phi(0)>1; \qquad \phi(1)>0 \tag{8.7}$$

The first part of (8.7) simply involves using (8.3) in (8.6). To check that $\phi(0)>1$, note that if $\phi(0)=[\delta d/2(a\delta+c)]\leqslant 1$, then, $\delta(d-2a)\leqslant 2c$, so that $[(b-2c)/(2a-b)]=\delta^*<\delta\leqslant[2c/(d-2a)]$, and this implies $d\leqslant[2b(a-c)/(b-2c)]$, a contradiction to our restriction on d. To check that $\phi(1)>0$, suppose on the contrary that $\delta d-b(1+\delta)\leqslant 0$. Then, we get $(b/d)\geqslant\delta/(1+\delta)>\delta^*/(1+\delta^*)=(b-2c)/2(a-c)$, so that $d\leqslant 2b(a-c)/(b-2c)$, which again contradicts our restriction on d.

Using (8.7) and $0<k_\delta<1$, there is $0<\underline{x}<k_\delta$ such that $\phi(\underline{x})=1$. This $\underline{x}$ is given by

$$\underline{x} = [\delta d-2(a\delta+c)]/b(1+\delta) \tag{8.8}$$

Defining $\bar{x}$ to be $\phi(1)$, we note that $\bar{x} = \phi(1) < \phi(k_\delta) = k_\delta$. Further, using (8.6),

$$\bar{x} = [\delta d - b(1+\delta)]/2(a\delta + c) \tag{8.9}$$

so that, using the equation $\bar{x} = \phi(1) = \phi(\phi(\underline{x}))$, $[b(1+\delta)/2(a\delta + c)] < 1$, and $1 > \delta > \delta^*$, we have $\bar{x} > \underline{x}$. To verify this last assertion, note that $\bar{x} = \phi(1) = \phi(\phi(\underline{x})) = \theta(1-\gamma) + \gamma^2 \underline{x}$, where $\theta = [\delta d/2(a\delta + c)]$ and $\gamma = b(1+\delta)/2(a\delta + c) < 1$ for $1 > \delta > \delta^*$. Thus if $\bar{x} \leqslant \underline{x}$, we must have $\theta(1-\gamma) \leqslant (1-\gamma^2)\,\underline{x} = (1-\gamma)(1+\gamma)\,\underline{x}$, and so $\underline{x} \geqslant \theta/(1+\gamma)$. But $\theta/(1+\gamma) = k_\delta$, so $\underline{x} = k_\delta$, a contradiction. This establishes that $\bar{x} > \underline{x}$.

Now, we can define

$$f(x) = \begin{cases} 1 & \text{for} \quad 0 \leqslant x \leqslant \underline{x} \\ \phi(x) & \text{for} \quad \underline{x} < x \leqslant 1 \end{cases} \tag{8.10}$$

and check, by using the Kuhn–Tucker theorem, that $f(x)$ solves (P) uniquely. For $\underline{x} < x \leqslant 1$, this is clear because $W_2(x, \phi(x)) = 0$, using (8.2) and (8.6). Thus, $W(x, z) - W(x, f(x)) \leqslant W_2(x, \phi(x))(z - f(x)) = 0$ for all $z \in X$. For $0 \leqslant x \leqslant \underline{x}$, we have $W_2(x, 1) = -bx(1+\delta) + \delta[d - 2a] - 2c \geqslant -b\underline{x}(1+\delta) + \delta[d - 2a] - 2c = -[\delta d - 2(a\delta + c)] + \delta[d - 2a] - 2c = 0$. Thus, for $0 \leqslant x \leqslant \underline{x}$, $W(x, z) - W(x, f(x)) \leqslant W_2(x, 1)(z - 1) \leqslant 0$ for all $z \in X$.

Using (8.10), we can obtain

$$F(x) = \begin{cases} \bar{x} & \text{for} \quad 0 \leqslant x \leqslant \underline{x} \\ \bar{x} + (x - \underline{x})\,\gamma^2 & \text{for } \underline{x} < x \leqslant 1 \end{cases} \tag{8.11}$$

Clearly, for $0 \leqslant x \leqslant \underline{x}$, $F(x) = F(\underline{x}) = \bar{x} > \underline{x} \geqslant x$, so there is no fixed point of F in this range of x. For $\underline{x} < x \leqslant 1$, $F(x) = \bar{x} + (x - \underline{x})\,\gamma^2$, so there is a unique fixed point, given by $k = [\bar{x} - \underline{x}\gamma^2]/(1-\gamma^2) = \theta(1-\gamma)/(1-\gamma^2) = \theta/(1+\gamma) = k_\delta$. (See Fig. 6 for graphs of f and F). This contradiction establishes that $0 < \delta < \delta^*$.

We now subdivide our analysis into three parts. Define

$$\delta_1 = 2c/(d - 2a); \; \delta_2 = b/(d - b) \tag{8.12}$$

We note that $0 < \delta_1 < \delta_2 < \delta^*$. Consider, first, the case in which $\delta_2 \leqslant \delta < \delta^*$. Here, we have

$$\phi(0) > 1, \qquad \phi(1) \geqslant 0, \qquad \phi(k_\delta) = k_\delta \tag{8.13}$$

Defining $\underline{x}$ and $\bar{x}$ by (8.8) and (8.9) respectively, we have $\bar{x} = \phi(1) < \phi(k_\delta) = k_\delta$, and furthermore, $\bar{x} \leqslant \underline{x}$. To verify this last inequality, note that if

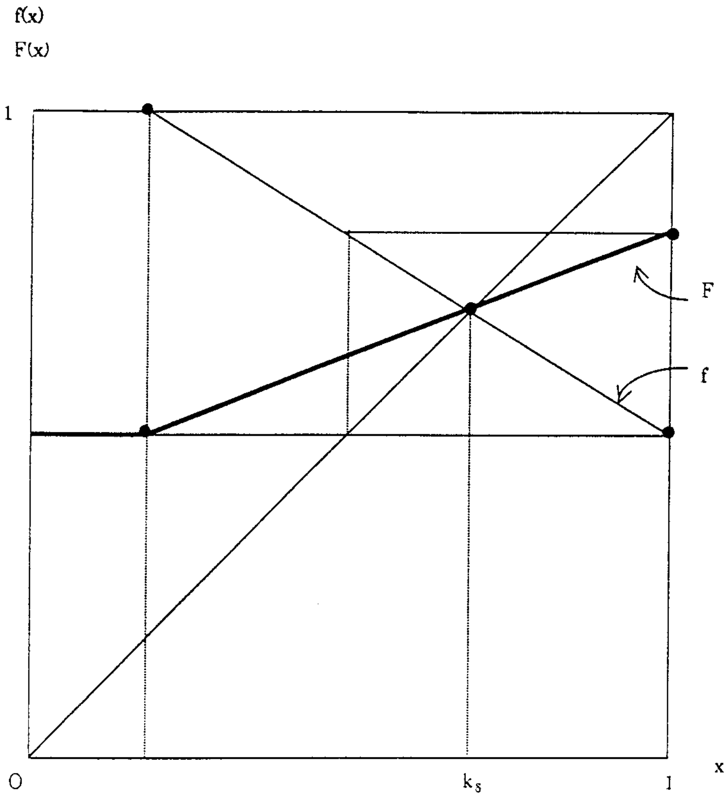

FIG. 6. Sutherland's example: Diagram of $F = f^2$ for $1 > \delta > \delta^*$.

$\bar{x} > \underline{x}$, then $\theta(\gamma - 1) = \gamma^2 \underline{x} - \bar{x} < (\gamma^2 - 1)\,\bar{x}$, and so $\underline{x} > \theta/(\gamma + 1) = k_\delta$, since $\gamma > 1$. This contradiction establishes that $\bar{x} \leqslant \underline{x}$.

We can now define $\tilde{x}$ such that $\phi(\tilde{x}) = \underline{x}$. Since $\phi(1) = \bar{x} \leqslant \underline{x}$ and $\phi(k_\delta) = k_\delta < 1 = \phi(\underline{x})$, so that $\underline{x} < k_\delta = \phi(k_\delta)$, $\tilde{x}$ is well-defined, and $k_\delta < \tilde{x} \leqslant 1$. Now defining

$$f(x) = \begin{cases} 1 & \text{for} \quad 0 \leqslant x \leqslant \underline{x} \\ \phi(x) & \text{for} \quad \underline{x} < x \leqslant 1 \end{cases} \tag{8.14}$$

we can check that f solves (P) uniquely. Furthermore, we get

$$F(x) = \begin{cases} \bar{x} & \text{for} \quad 0 \leqslant x \leqslant \underline{x} \\ \bar{x} + \gamma^2 (x - \underline{x}) & \text{for} \quad \underline{x} < x < \tilde{x} \\ 1 & \text{for} \quad \tilde{x} \leqslant x \leqslant 1 \end{cases} \tag{8.15}$$

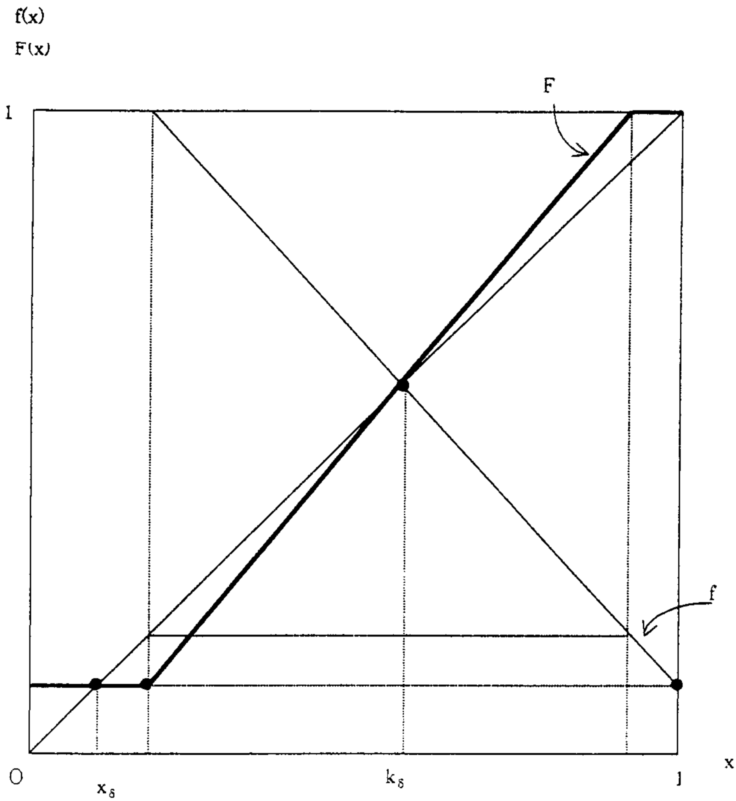

FIG. 7. Sutherland's example: Diagram of $F=f^2$ for $\delta_2 \leqslant \delta < \delta^*$.

The fixed points of F are $\bar{x}, k_\delta$ and 1, and $F'(k_\delta)=\gamma^2>1$. (For graphs of f and F, see Fig. 7).

Next, we consider the case in which $\delta_1 \leqslant \delta < \delta_2$. Here we have

$$\phi(0) \geqslant 1, \qquad \phi(1) < 0, \qquad \phi(k_\delta) = k_\delta \tag{8.16}$$

Defining $\underline{x}$ by (8.8), x' by $\phi(x')=0$, and $\tilde{x}$ by $\phi(\tilde{x})=\underline{x}$, we note that $0 \leqslant \underline{x} < k_\delta < \tilde{x} < x' < 1$. Now, if we define x'' by $\phi(x'')=x'$, we can check that $\underline{x} < x'' < k_\delta$. Then, we can define

$$f(x) = \begin{cases} 1 & \text{for} \quad 0 \leqslant x \leqslant \underline{x} \\ \phi(x) & \text{for} \quad \underline{x} < x < x' \\ 0 & \text{for} \quad x' \leqslant x \leqslant 1 \end{cases} \tag{8.17}$$

and verify that this f solves (P) uniquely. Furthermore, we obtain

$$F(x)=\begin{cases}0 & \text{for} \quad 0\leqslant x\leqslant x'' \\ \gamma^2(x-x'') & \text{for} \quad x''<x<\tilde{x} \\ 1 & \text{for} \quad \tilde{x}\leqslant x\leqslant 1\end{cases} \tag{8.18}$$

The fixed points of F are 0, k_δ and 1, and $F'(k_\delta)=\gamma^2>1$. (For graphs of f and F, see Fig. 8).

Finally, we consider the case in which $0<\delta<\delta_1$. Here, we have

$$0<\phi(0)<1, \qquad \phi(1)<0, \qquad \phi(k_\delta)=k_\delta \tag{8.19}$$

Defining x' by $\phi(x')=0$, and x'' by $\phi(x'')=x'$, we can check that $0<x''<k_\delta<x'<\theta=\phi(0)<1$.

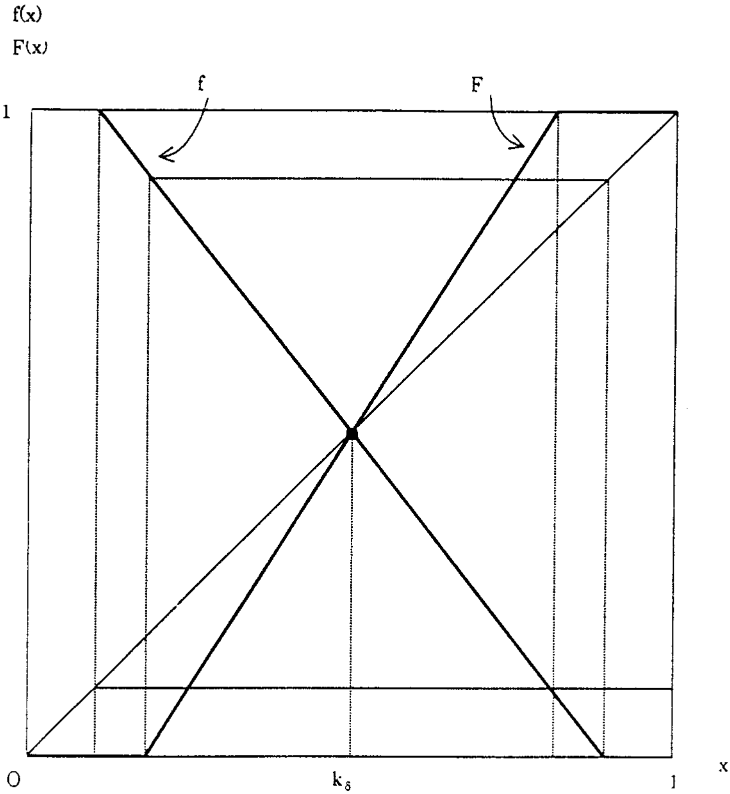

FIG. 8. Sutherland's example: Diagram of $F=f^2$ for $\delta_1\leqslant\delta<\delta_2$.

Then, we can define

$$f(x)=\begin{cases}\phi(x) & \text{for} \quad x<x<x' \\ 0 & \text{for} \quad x'\leqslant x\leqslant 1\end{cases} \tag{8.20}$$

and verify that this f solves (P) uniquely. Furthermore, we have

$$F(x)=\begin{cases}0 & \text{for} \quad 0\leqslant x\leqslant x'' \\ \gamma^2(x-x'') & \text{for} \quad x''<x<x' \\ \theta & \text{for} \quad x'\leqslant x\leqslant 1\end{cases} \tag{8.21}$$

The fixed points of F are 0, k_δ, and θ, and $F'(k_\delta)=\gamma^2>1$. (The graphs of f and F are shown in Fig. 9). This completes our verification of Condition HI.

Condition US can be verified as follows. The numerator of $R(x)$ is $(bx+2cx)\,2a+(d-2ax-bx)\,2c$, while the denominator of $R(x)$ is

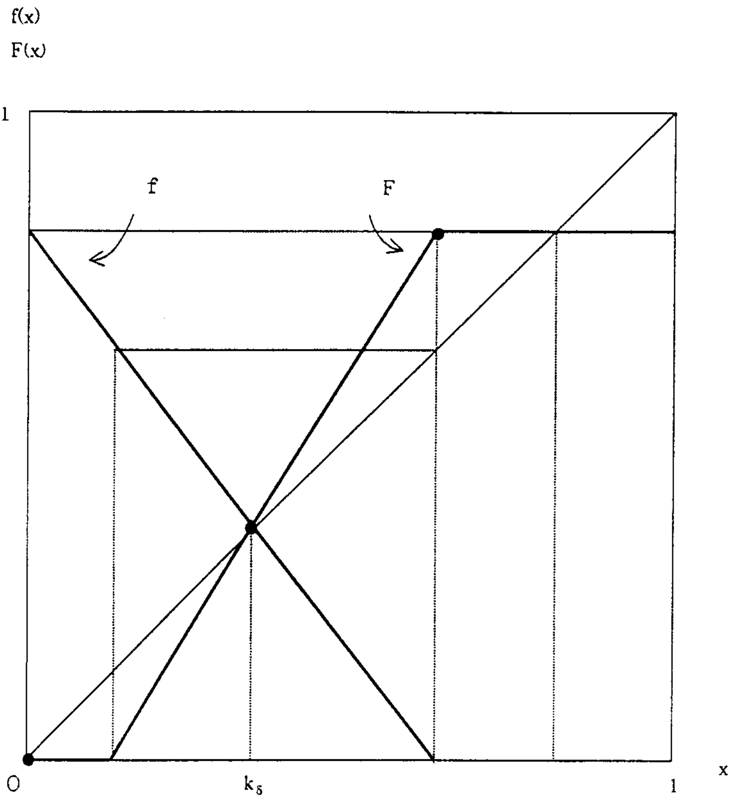

FIG. 9. Sutherland's example: Diagram of $F=f^2$ for $0<\delta<\delta_1$.

$\{(d-2ax-bx)+(bx+2cx)\}$ b. Thus, we get $R(x)=[2cd-2bcx+2abx]/[db-2abx+2bcx]=[2cd+2bx(a-c)]/[db-2bx(a-c)]$. As x increases, $2bx(a-c)$ increases since $a>c$. Thus, $R(x)$ increases as x increases. Also, $\lim_{x\to 0} R(x)=(2c/b)\in(0,1)$.

Our analysis above yields the global bifurcation diagram shown in Fig. 10. For $1>\delta>\delta^*$, the stationary optimal stock, k_δ, is globally stable, and there are no other periodic points. For $\delta_2\leqslant\delta<\delta^*$, the stationary optimal stock, k_δ is locally unstable, and there are two periodic points, $\bar{x}$ and 1. For all $x\in X, x\neq k_\delta$, the optimal program $(x_t)_0^\infty$ from x has $\bar{x}$ and 1 as its limit points. Further, as δ decreases to $\delta_2, \bar{x}$ decreases to 0. For $\delta_1\leqslant\delta<\delta_2$, the stationary optimal stock, k_δ, is locally unstable, and there are two periodic points, 0 and 1, which are also the limit points of optimal programs from all $x\in X, x\neq k_\delta$. Finally, for $0<\delta<\delta_1$, the stationary optimal stock, k_δ, is locally unstable, and there are two periodic points, 0 and θ, which are also the limit points of optimal programs from all $x\in X, x\neq k_\delta$. Further, as δ decreases to 0, θ decreases to 0.

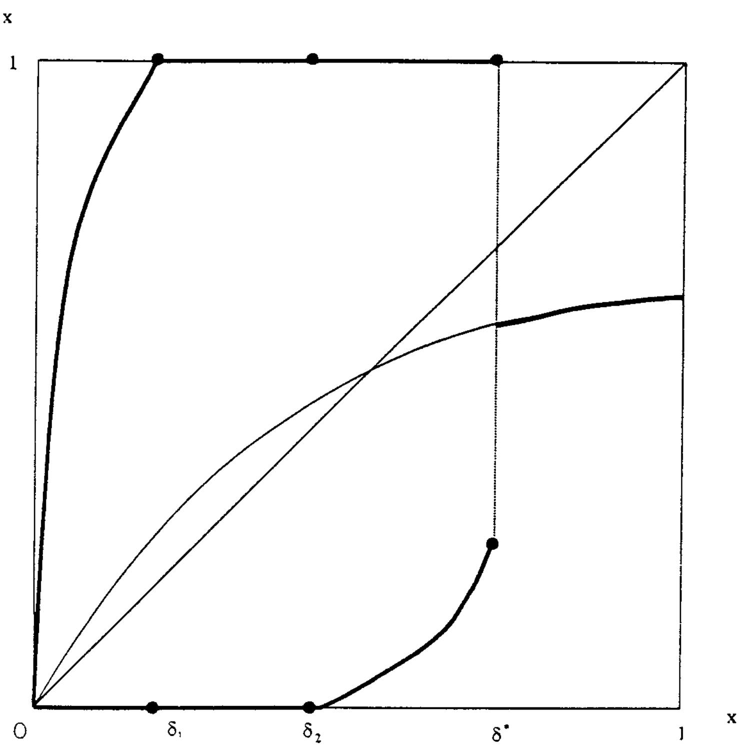

FIG. 10. Sutherland's example: Bifurcation diagram.

APPENDIX

The purpose of this appendix is to show that each bilateral Dini derivative of h, at the unique stationary optimal stock k, equals one of the roots of the characteristic equation (5.5) [associated with the Ramsey–Euler equation (5.1) at $x=k$].

Before proving the above result, we note that if $(x, h(x))$ is in int Ω, and $x \neq k$, then

$$\begin{aligned}\tilde{u}_{21} + [(-\tilde{u}_{22}) + \delta(-\bar{u}_{11})]\{[h(x) - h(k)]/(k-x)\} \\ -\delta(-\bar{u}_{12})\{[h^2(k) - h^2(x)]/(k-x)\} = 0 \qquad \text{(A.1)}\end{aligned}$$

where $\tilde{u}_{21}$ and $\tilde{u}_{22}$ are evaluated at an appropriate convex combination of $(x, h(x))$ and (k, k), and $\bar{u}_{11}$ and $\bar{u}_{12}$ are evaluated at an appropriate convex combination of $(h(x), h^2(x))$ and (k, k), as given by the Mean-Value theorem. To see this, write the Ramsey–Euler equations:

$$u_2(k, k) + \delta u_1(k, k) = 0; \qquad u_2(x, h(x)) + \delta u_1(h(x), h^2(x)) = 0$$

Use the Mean Value theorem to get

$$\begin{aligned}\tilde{u}_{21}(k-x) + (-\tilde{u}_{22})(h(x)-k) + \delta(-\bar{u}_{11})(h(x)-k) \\ -\delta(-\bar{u}_{12})[k - h^2(x)] = 0\end{aligned}$$

Dividing by $(k-x) \neq 0$, we obtain (A.1).

Let us define

$$\mathbf{m} = \min\left[\liminf_{x \to k+} \left|\frac{h(x)-h(k)}{x-k}\right|, \liminf_{x \to k-} \left|\frac{h(x)-h(k)}{x-k}\right|\right]$$

$$\mathbf{M} = \max\left[\limsup_{x \to k+} \left|\frac{h(x)-h(k)}{x-k}\right|, \limsup_{x \to k-} \left|\frac{h(x)-h(k)}{x-k}\right|\right]$$

Let λ_1 and λ_2 be the roots of the characteristic equation associated with the Ramsey–Euler equation (5.1). Then λ_1, λ_2 are real and negative. We claim that

(i) $\mathbf{M}$ is either $(-\lambda_1)$ or $(-\lambda_2)$; (ii) $\mathbf{m}$ is either $(-\lambda_1)$ or $(-\lambda_2)$. (A.2)

We will establish only (i), since the proof of (ii) is similar. Let us define:

$$g(\lambda) = (-u_{21}(k, k)) + (-u_{22}(k, k))\,\lambda + \delta(-u_{11}(k, k))\,\lambda + \delta(-u_{12}(k, k))\,\lambda^2$$

Claim 1. $g(-\mathbf{M})>0$ is not possible. For if $g(-\mathbf{M})>0$, then

$$(-u_{21}(k,k))+[(-u_{22}(k,k))+\delta(-u_{11}(k,k))](-\mathbf{M}) \\ +\delta(-u_{12}(k,k))\,\mathbf{M}^2>0$$

We can choose $\varepsilon>0$ such that

$$(-u_{21}(k,k))>[(-u_{22}(k,k))+\delta(-u_{11}(k,k))](\mathbf{M}+\varepsilon) \\ -\delta(-u_{12}(k,k))(\mathbf{M}^2-\varepsilon^2)$$

Given $\varepsilon>0$, one can find $z^s\to k$ $(z^s\neq k)$ such that

$$[h(k)-h(z^s)]/(z^s-k)\geqslant(\mathbf{M}-\varepsilon) \tag{A.3}$$

One can then find S' and x^s such that $h(x^s)=z^s$ for $s\geqslant S'$. Then $x^s\to k$ $(x^s\neq k)$ as $s\to\infty$, and one can find $S\geqslant S'$, such that for $s\geqslant S$,

$$[h(x^s)-h(k)]/(k-x^s)\leqslant(\mathbf{M}+\varepsilon) \tag{A.4}$$

Using (A.1) with $x=x^s$, (A.3) and (A.4), we have

$$\begin{aligned}(-\tilde{u}_{21})&=\left[(-\tilde{u}_{22})+\delta(-\bar{u}_{11})-\delta(-\bar{u}_{12})\left(\frac{h(z^s)-h(k)}{k-z^s}\right)\right]\left[\frac{h(x^s)-h(k)}{k-x^s}\right]\\ &\leqslant[(-\tilde{u}_{22})+\delta(-\bar{u}_{11})-\delta(-\bar{u}_{12})(\mathbf{M}-\varepsilon)]\left[\frac{h(x^s)-h(k)}{k-x^s}\right]\\ &\leqslant[(-\tilde{u}_{22})+\delta(-\bar{u}_{11})-\delta(-\bar{u}_{12})(\mathbf{M}-\varepsilon)](\mathbf{M}+\varepsilon)\end{aligned}$$

Letting $x^s\to k$, we get $(-u_{21}(k,k))\leqslant[(-u_{22}(k,k))+\delta(-u_{11}(k,k))-\delta(-u_{12}(k,k))(\mathbf{M}-\varepsilon)](\mathbf{M}+\varepsilon)<(-u_{21}(k,k))$, a contradiction, which establishes the claim.

Claim 2. $g(-\mathbf{M})<0$ is not possible. Otherwise,

$$(-u_{21}(k,k))+[(-u_{22}(k,k))+\delta(-u_{11}(k,k))](-\mathbf{M}) \\ +\delta(-u_{12}(k,k))\,\mathbf{M}^2<0$$

We can choose $\varepsilon>0$ such that

$$(-u_{21}(k,k))<[(-u_{22}(k,k))+\delta(-u_{11}(k,k))](\mathbf{M}-\varepsilon) \\ -\delta(-u_{12}(k,k))(\mathbf{M}^2-\varepsilon^2)$$

One can find $x^s\to k$ $(x^s\neq k)$ such that

$$[h(k)-h(x^s)]/(x^s-k)\geqslant(\mathbf{M}-\varepsilon) \tag{A.5}$$

Define $z^s = h(x^s)$. Then $z^s \to k$ ($z^s \neq k$) and we can find S, such that for $s \geqslant S$,

$$[h(k) - h(z^s)]/(z^s - k) \leqslant (\mathbf{M} + \varepsilon) \tag{A.6}$$

Using (A.1) with $x = x^s$, (A.5) and (A.6), we have

$$(-\tilde{u}_{21}) = \left[(-\tilde{u}_{22}) + \delta(-\bar{u}_{11}) - \delta(-\bar{u}_{12}) \left[\frac{h(k) - h(z^s)}{z^s - k} \right] \right] \left[\frac{h(x^s) - h(k)}{k - x^s} \right]$$
$$\geqslant [(-\tilde{u}_{22}) + \delta(-\bar{u}_{11}) - \delta(-\bar{u}_{12})(\mathbf{M} + \varepsilon)](\mathbf{M} - \varepsilon)$$

Letting $x^s \to k$, we get

$$(-u_{21}(k, k)) \geqslant [(-u_{22}(k, k)) + \delta(-u_{11}(k, k))](\mathbf{M} - \varepsilon)$$
$$-\delta(-u_{12}(k, k))(\mathbf{M}^2 - \varepsilon^2) > (-u_{21}(k, k)),$$

a contradiction. From Claims 1 and 2, we get $g(-\mathbf{M}) = 0$, so $\mathbf{M} = (-\lambda_1)$ or $(-\lambda_2)$.

REFERENCES

1. A. Araujo, The once but not twice differentiability of the policy function, *Econometrica* **59** (1991), 1383–1393.
2. A. Araujo and J. A. Scheinkman, Smoothness, comparative dynamics, and the turnpike property, *Econometrica* **45** (1977), 601–620.
3. J. Benhabib and K. Nishimura, Competitive equilibrium cycles, *J. Econ. Theory* **35** (1985), 284–306.
4. L. M. Benveniste and J. A. Scheinkman, On the differentiability of the value function in dynamic models of economics, *Econometrica* **47** (1979), 727–732.
5. M. Boldrin, Paths of optimal accumulation in a two-sector model, *in* "Economic Complexity: Chaos, Sunspots, Bubbles and Nonlinearity" (W. Barnett *et al.*, Eds.), Cambridge University Press, Cambridge, UK, 1988.
6. M. Boldrin and R. Deneckere, Sources of complex dynamics in two-sector growth models, *Econ. Dynamics Control* **14** (1990), 627–653.
7. M. Boldrin and L. Montrucchio, On the indeterminacy of capital accumulation paths, *J. Econ. Theory* **40** (1986), 26–39.
8. W. A. Brock and J. Scheinkman, Global asymptotic stability of optimal control systems with applications to the theory of economic growth, *J. Econ. Theory* **12** (1976), 164–190.
9. D. Cass and K. Shell, The structure and stability of competitive dynamical systems, *J. Econ. Theory* **12** (1976), 31–70.
10. P. Collet and J. Eckmann, "Iterated Maps on the Interval as Dynamical Systems," Birkhäuser, Boston, 1980.
11. W. De Melo and S. van Strien, "One-Dimensional Dynamics," Springer-Verlag, Berlin, 1993.
12. R. Deneckere and S. Pelikan, Competitive Chaos, *J. Econ. Theory* **40** (1986), 13–25.
13. P. K. Dutta and T. Mitra, Maximum theorems for convex structures with an application to the theory of optimal intertemporal allocation, *J. Math. Econ.* **18** (1989), 77–86.

14. J. Guckenheimer, Sensitive dependence to initial conditions for one dimensional maps, *Commun. Math. Phys.* **70** (1979), 133–160.
15. M. Majumdar and T. Mitra, Periodic and chaotic programs of optimal intertemporal allocation in an aggregative model with wealth effects, *Econ. Theory.* **4** (1994), 649–676.
16. L. W. McKenzie, A primal route to the turnpike and Liapounov stability, *J. Econ. Theory.* **27** (1982), 194–209.
17. L. McKenzie, Turnpike theory, discounted utility, and the Von Neumann facet, *J. Econ. Theory.* **30** (1983), 330–352.
18. L. W. McKenzie, Optimal economic growth, turnpike theorems and comparative dynamics, *in* "Handbook of Mathematical Economics" (K. J. Arrow and M. D. Intriligator, Eds.), Vol. III, North-Holland, Amsterdam, 1986.
19. M. Misiurewicz, Absolutely continuous measures for certain maps of an interval, *Publ. Math.* **53** (1981), 17–51.
20. T. Mitra, An exact discount factor restriction for period-three cycles in dynamic optimization models, *J. Econ. Theory* **69** (1996), 281–305.
21. L. Montrucchio, Thompson metric, contraction property and differentiability of policy functions, *J. Econ. Behavior Organization* **33** (1998), 449–466.
22. K. Nishimura, G. Sorger, and M. Yano, Ergodic chaos in optimal growth models with low discount rates, *Econ. Theory* **4** (1994), 705–717.
23. K. Nishimura and M. Yano, Optimal chaos, nonlinearity and feasibility conditions, *Econ. Theory* **4** (1994), 689–704.
24. K. Nishimura and M. Yano, Nonlinear dynamics and optimal growth: An example, *Econometrica* **63** (1995), 981–1001.
25. K. Nishimura and M. Yano, Nonlinearity and business cycles in a two-sector equilibrium model: An example with Cobb–Douglas production functions, *in* "Nonlinear and Convex Analysis in Economic Theory" (T. Maruyama and W. Takahashi, Eds.), Springer-Verlag, Berlin, 1995.
25. K. Nishimura and M. Yano, On the least upper bound of discount factors that are compatible with optimal period-three cycles, *J. Econ. Theory* **69** (1996), 306–333.
27. S. M. Ross, "Introduction to Stochastic Dynamic Programming," Academic Press, New York, 1983.
28. P. A. Samuelson, "Foundations of Economic Analysis," Harvard University Press, Cambridge, MA, 1947.
29. P. A. Samuelson, Optimality of profit, including prices under ideal planning, *Proc. Nat. Acad. Sci.* **70** (1973), 2109–2111.
30. M. S. Santos, Smoothness of the policy function in discrete-time economic models, *Econometrica* **59** (1991), 1365–1382.
31. J. A. Scheinkman, On optimal steady states of *n*-sector growth models when utility is discounted, *J. Econ. Theory* **12** (1976), 11–30.
32. D. Singer, Stable orbits and bifurcations of maps of the interval, *SIAM J. Appl. Math.* **35** (1978), 260–267.
33. G. Sorger, On the minimum rate of impatience for complicated optimal growth paths, *J. Econ. Theory* **12** (1992), 11–30.
34. G. Sorger, "Minimum Impatience Theorems for Recursive Economic Models," Springer-Verlag, Berlin, 1992.
35. N. L. Stokey, R. E. Lucas, Jr., and E. C. Prescott, "Recursive Methods in Economic Dynamics," Harvard University Press, Cambridge, MA, 1989.
36. W. R. S. Sutherland, On optimal development in multi-sectoral economy: The discounted case, *Rev. Econ. Stud.* **46** (1970), 585–589.
37. D. Topkis, Minimizing a submodular function on a lattice, *Oper. Res.* **26** (1968), 305–321.